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# Amending an inconsistency in JCGM 100:2008 (GUM 1995 with minor corrections)

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# Amendment to the legacy GUM

- An amendment to the legacy GUM, i.e. JCGM 100:2008, alias GUM 1995 with minor corrections, is circulating among the JCGM Member Organisations (Mos)

- This is a first in thirty years, so that it is expected to be solidly motivated

- It is intended to amend an inconsistency (subtly hidden)

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# Background

- JCGM 100:2008

Measurement model

$$Y = f(X_1, X_2, \dots, X_N)$$

Uncorrelated case

$$u^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i)$$

# Background

- When the nonlinearity of the measurement model is significant, higher-order terms in the Taylor expansion should be taken into account. For normal, uncorrelated input variables

$$\sum_{1=1}^{N} \sum_{j=1}^{N} \left[ \frac{1}{2} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + \frac{\partial f}{\partial x_i} \frac{\partial^3 f}{\partial x_i \partial x_j^2} \right] u^2(x_i) u^2(x_j) \qquad (1)$$

- This is legacy GUM, Note to 5.1.2

- No mention about the impact on the estimate $y$

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# Background

- The impact of non linearity on the estimate is discussed

- ambiguously in a Note to 4.1.4

$$y = \bar{Y} = \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{n}\sum_{k=1}^{n} f\left(X_{1,k}, X_{2,k}, \dots, X_{N,k}\right) \qquad (2)$$

- correctly, but with a misleading heading (Asymmetric distributions of possible values) in F.2.4.4

$$y = f(x_1, x_2, \dots, x_N) + \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2} u^2(x_i) \qquad (3)$$

# Background

- In example H.1, End gauge calibration, correction (1) to the uncertainty is discussed and applied

- In examples

- H.2, Simultaneous resistance and reactance measurement
- H.4, Measurement of activity

- Expression (2) is compared with the usual $y = f\ (x_1, x_2, \ldots, x_N)$

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# Background

- In example H.2, expression

$$y = f(x_1, x_2, \dots, x_N) + \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j} u(x_i, x_j)$$

is given. This is a generalisation of expression (3)

$$y = f(x_1, x_2, \dots, x_N) + \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2} u^2(x_i) \qquad (3)$$

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# The amendment

- The amendment simply states that when higher-order terms in the Taylor expansion are considered, they impact not only on the uncertainty, but also on the estimate, inserting where appropriate expression (3)

$$y = f(x_1, x_2, \ldots, x_N) + \frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2} u^2(x_i) \qquad (3)$$

- Example H.1, End gauge calibration, needs as well to be amended to include the discussion about the estimate

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# Amendment to the legacy GUM

- The amendment is accompanied by a «Motivation»

# Example

- $\delta Y = X_1^2 + X_2^2$
- $X_1, X_2$ are dimensionless and uncorrelated
- Three cases:

$x_1 = x_2 = 0,\qquad x_1 = 0.010,\ x_2 = 0$ and $x_1 = 0.050,\ x_2 = 0$

- $u(x_1) = u(x_2) = 0.005$

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# •Case 1

- $x_1 = x_2 = 0,$ $u(x_1) = u(x_2) = 0.050$
- Using first-order terms only ($GUF_1$)
- $\delta y = 0$ $u(\delta y) = 0$!!! (first derivatives are identically zero)
- Using second-order terms ($GUF_2$ – higher order derivatives are zero)
- $\delta y = 50 \times 10^{-6}$!!! $u(\delta y) = 50 \times 10^{-6}$
- Coinciding with analytic solution and Monte Carlo Method (MCM)

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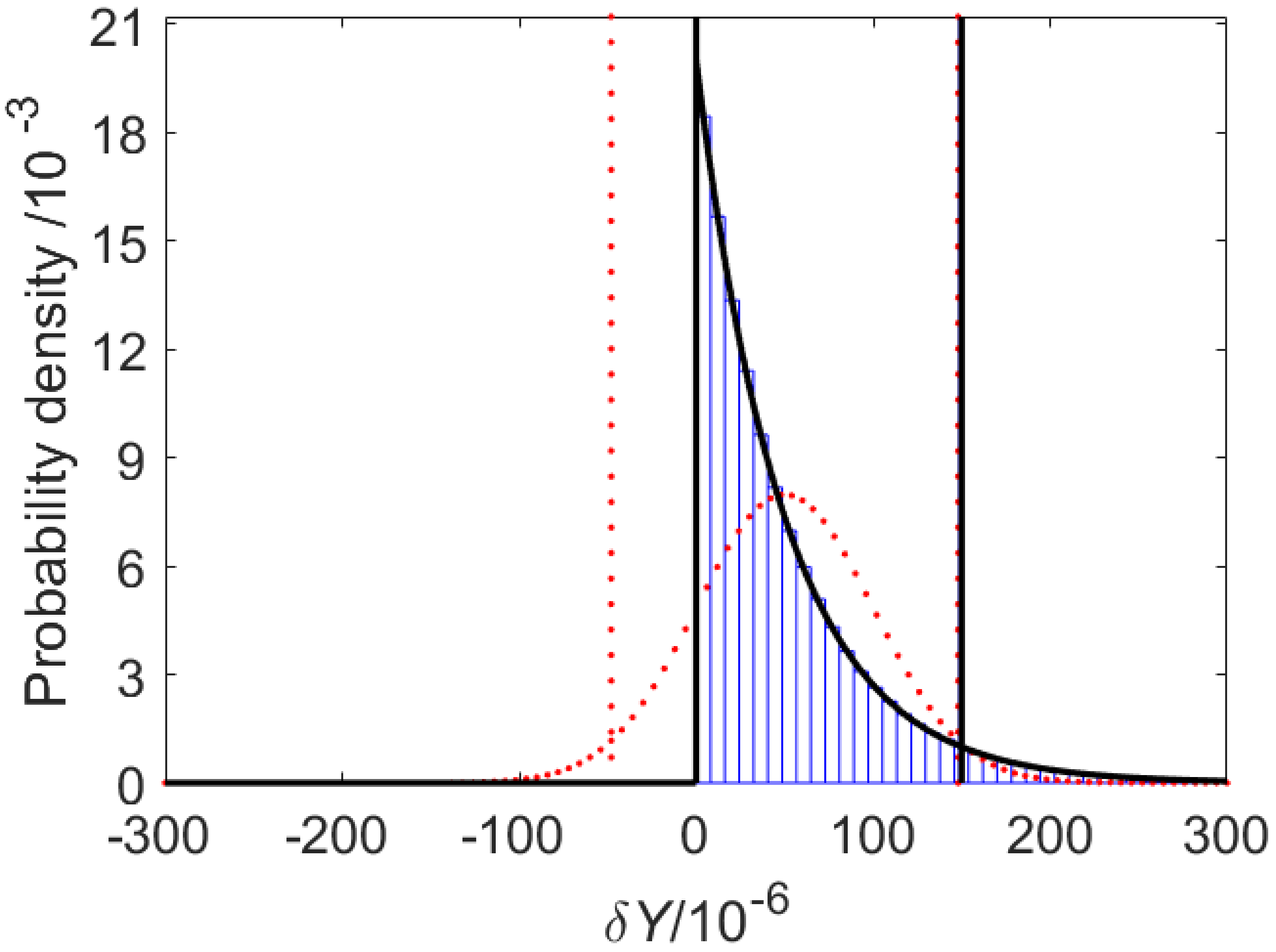
# Case 1

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# •Case 2

- $x_1 = 0.010,\ x_2 = 0$ $u(x_1) = u(x_2) = 0.005$

- GUF$_1$
- $\delta y = 100 \times 10^{-6}$ $u(\delta y) = 100 \times 10^{-6}$

- GUF$_2$
- $\delta y = 150 \times 10^{-6}$ $u(\delta y) = 112 \times 10^{-6}$

- GUF$_2$ coincides with MCM and analytic solution

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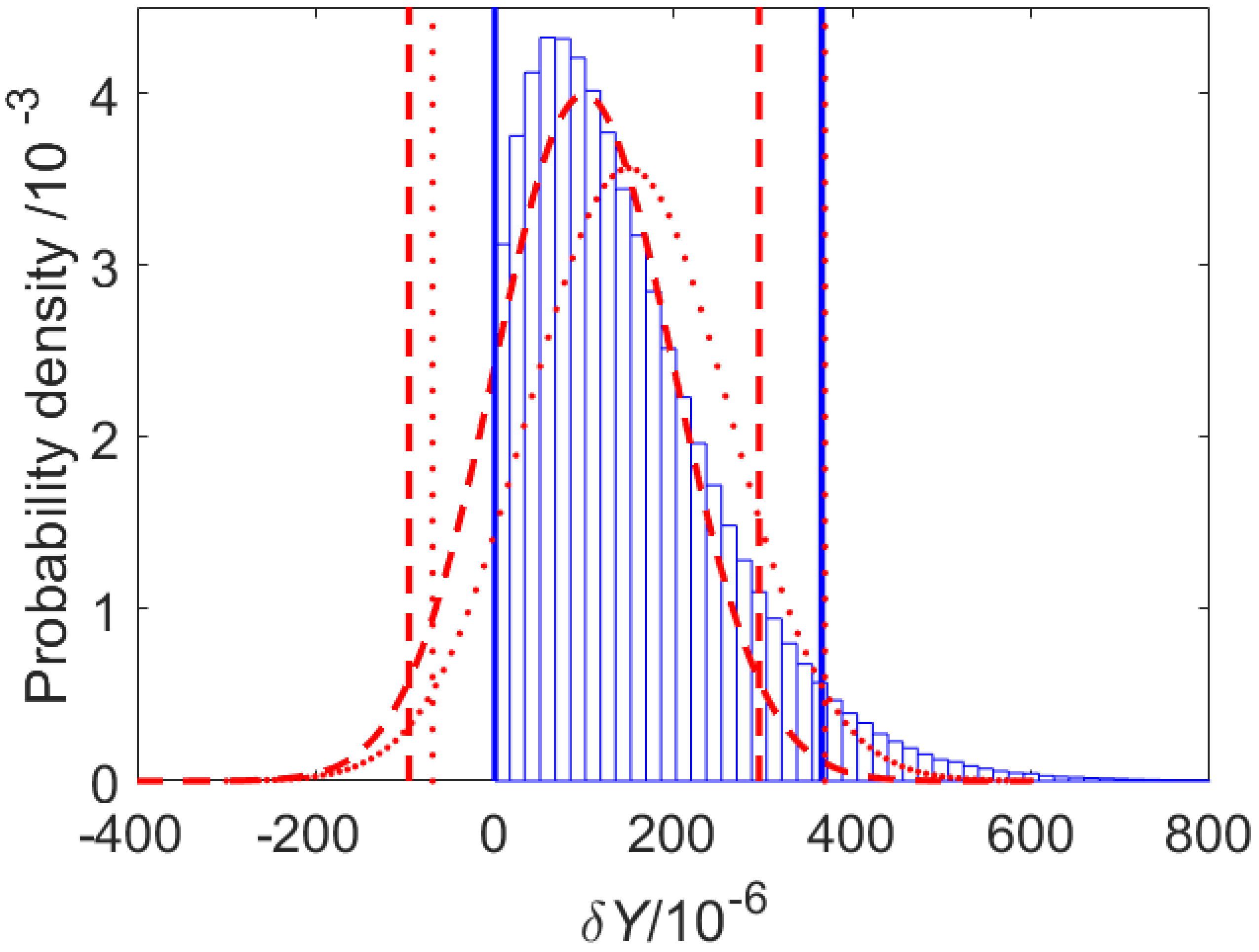
# Case 2


Probability density /10^-3
0
1
2
3
4
-400
-200
0
200
400
600
800
δY/10^-6

# •Case 3

- $x_1 = 0.050, \; x_2 = 0$ $\qquad u(x_1) = u(x_2) = 0.005$

- $\text{GUF}_1$
- $\delta y = 2500 \times 10^{-6}$ $\qquad u(\delta y) = 500 \times 10^{-6}$

- $\text{GUF}_2$
- $\delta y = 2550 \times 10^{-6}$ $\qquad u(\delta y) = 502 \times 10^{-6}$

- Again, $\text{GUF}_2$ coincides with MCM and analytic solution

- $\text{GUF}_1$ behaves here much better than in cases 1 and 3. First-order terms become dominant and the CLT holds.

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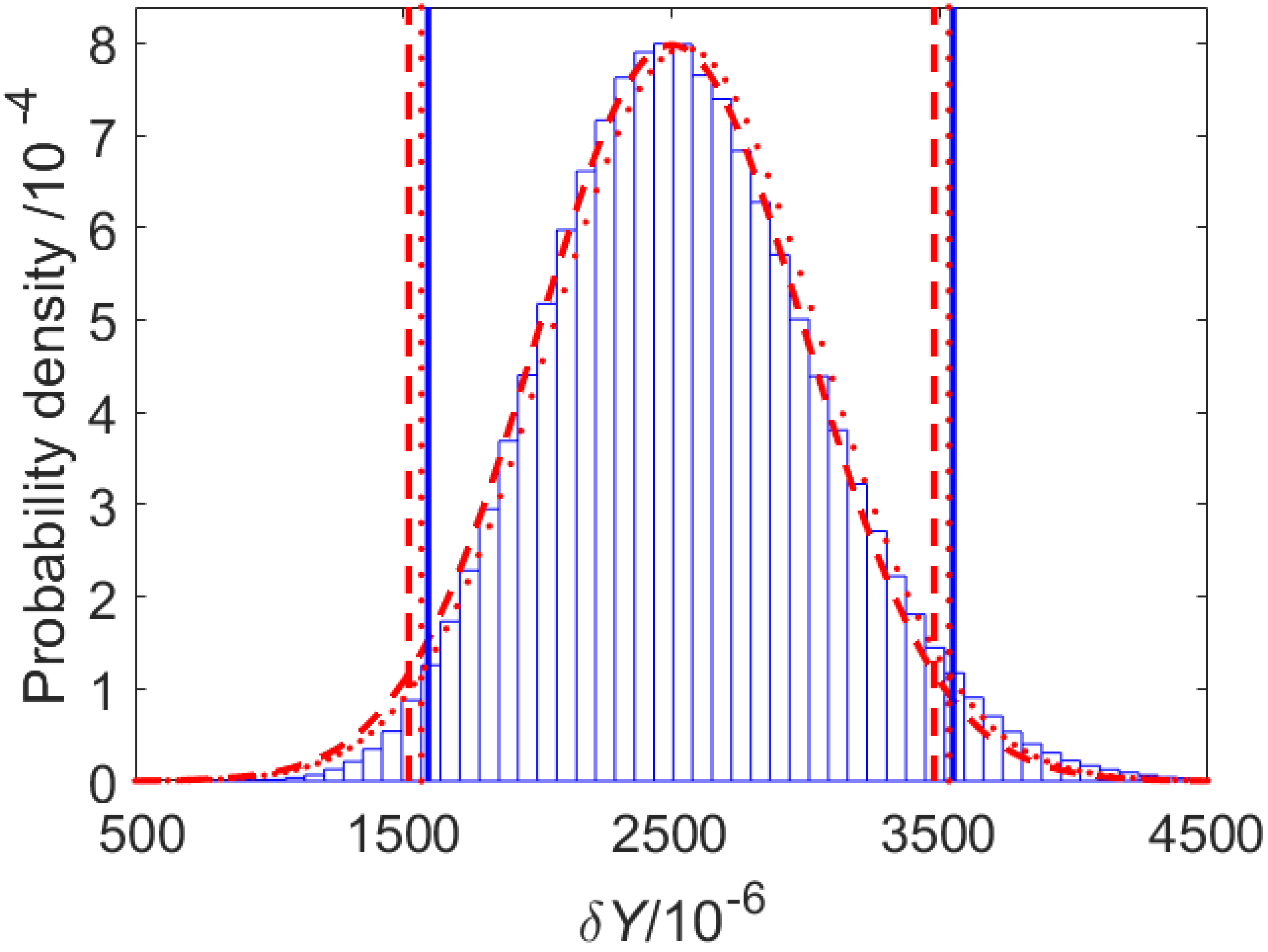
# Case 3



Probability density /10^-4
8
7
6
5
4
3
2
1
0
500
1500
2500
3500
4500
δY/10^-6

# Key takeaways

- Model $\delta Y = X_1^2 + X_2^2$ is an algebraic relationship among physical quantities
- In a purely algebraic context, $0 + 0 = 0$
- However, the same model is as well a relationship among random variables describing state of knowledge about the quantities involved
- In the latter context, $E(X_1) = E(X_2) = 0$ yields $E(\delta Y) \neq 0$
- The legacy GUM needs to explicitly acknowledge this fact and user need to accept it

- THE END