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A Bayesian model for measurements by counting

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A Bayesian model for measurements by counting

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Abstract Counting processes occur very often in several scientific and technological problems. The concept of numerousness and, consequently, the counting of a number of items are at the base of many high-level measurements, as well as in everyday life applications. The occurrence of under/over-counting errors is real and needs to be addressed. The present work proposes a Bayesian model able to incorporate prior information on the actual number of items to be counted and the available knowledge on the counting errors.

Keywords: Measurement by counting, Bayesian Model, Counting errors, Uncertainty

1. Introduction

Counting objects or events (hereafter simply "items") occurs very often in many scientific and technological domains, being at the base of many high-level measurements in fields such as, for example, time and frequency, optics, ionizing radiations, microbiology and chemistry. Also in everyday life countings play a fundamental role, as for the use of electric power meters, based on counting a number of impulsions, the assessment of pre-packaged goods, whose total price depends on the number of exchanged items, and the monitoring of medication consumption, for example by monitoring of pill intake into blisters.

As for any measurement, the result of a counting should be expressed as: a) an estimate (the number of counted items) of the measurand (the true number of items) and an associated uncertainty (or a suitable coverage interval), according to the "Guide to the expression of uncertainty in measurement" (GUM) [1]; or, b) a probability distribution (discrete, in the case of counting) describing the state of knowledge on the measurand, according to Supplement 1 to the GUM [2]. It is to be noted, however, that neither of the above-mentioned documents provides guidance on the issue of uncertainty in measurement by counting.

In spite of a large number of publications available on the counting topic in so many different fields and applications, we could hardly find studies aimed at correcting counting errors in the measurand estimate and evaluating the associated uncertainty. This motivated us to work on such specific need.

2. Aim of the work and notation

The count (or estimate) of a number Y of items (the measurand) can be wrong because of human or instrumental errors. It might occur, for example, that one fails in counting an item. In case of such false negative errors, the measurand is underestimated. On the other hand, one may count a non-existing item (or count twice an existing item), hence making a false positive error and overestimating the measurand. So, given an observed value or an estimate x for the measurand, it is needed to know the probability that $\{Y = x\}$, or $\{Y = x+1\}$, $\{Y = x-1\}$ and so on. In other words, one seeks the probability mass function (pmf) that can be associated to the measurand Y on the basis of the available information.

In a previous paper [3], we proposed a general model for measurements by counting to obtain an estimate and an associated standard uncertainty, as well as a pmf for the measurand. That model, however, can accommodate only missing and double countings and is not able to take into account any prior knowledge on the measurand. In the present work we tackle the problem using a fully Bayesian model, allowing the determination of a posterior pmf for the measurand, taking into account the probability of errors of over and under counting, i.e., of false positives and negatives, and correcting the estimate of the measurand for the bias due to such errors.

In our scenario, we consider a finite population made of n possible items, and distinguish between the number Y of existing items, i.e. the measurand, and the number $n - Y$ of non-existing items. For example, the passengers in an airplane are the existing items and their number the measurand Y, whereas the empty seats constitute the non-existing items, the population being the total number of seats n, which is thus the maximum value that the measurand can have, i.e., $Y \leq n$. Following the notation in [1], Y denotes both the measurand and the random variable describing the state of knowledge about it. The (integer) values that Y can take are denoted by y $(0 \leq y \leq n)$. A count for Y is denoted by x, X being the random variable of which x is a realization $(0 \leq x \leq n)$.

The probabilities that an item exists or does not exist are denoted by $p_\uparrow$ and $p_\downarrow = 1 - p_\uparrow$, respectively. As regards counts, $p_\leftarrow$ is the (conditional) probability that a non-existing item is mistakenly counted (probability of false positive), and $p_\rightarrow$ is the (conditional) probability that an existing item is correctly counted (complementary of the probability of false negative).

Resorting to communication theory [4], we see the (sequential) counting of items as a discrete system of two steps: 1) the transmission of a sequence of discrete signals, e.g., 1_T and 0_T (the subscript “T” standing for “transmitted”), coding the presence and the absence of an item, respectively; 2) the reception of a corresponding sequence of discrete signals, e.g., 1_R and 0_R, indicating that an item has been counted or not, respectively (the subscript “R” standing for “received”). According to this model, depicted in Fig. 1, each event in the top sequence of symbols is characterized by an occurrence probability $p_\uparrow$. In the airplane example, $p_\uparrow = P(1_\mathrm{T})$ is the probability that a seat is occupied, the probability of an empty seat thus being $p_\downarrow = 1 - p_\uparrow$. Then, the transmitted signal may be perturbed by noise, hence leading to a different received signal. This accounts for the ability of the counter to correctly count a transmitted 1_T (described by a conditional probability $p_\rightarrow = P(1_\mathrm{R}|1_\mathrm{T})$) and to correctly disregard a 0_T. We define the probability $P(0_\mathrm{R}|0_\mathrm{T})$ of the latter case as $1 - p_\leftarrow$, where $p_\leftarrow = P(1_\mathrm{R}|0_\mathrm{T})$ is the probability that a non-occurring event is counted (an empty seat erroneously counted as occupied or, equivalently, a double counting of the same passenger). Similarly, $1 - p_\rightarrow = P(0_\mathrm{R}|1_\mathrm{T})$ is the probability of missing an existing item. Each symbol is transmitted independently of the others and generates the second level as a corresponding sequence of 0_R and 1_R. Random variables Y and X are associated with the total number of 1_T and 1_R, respectively.

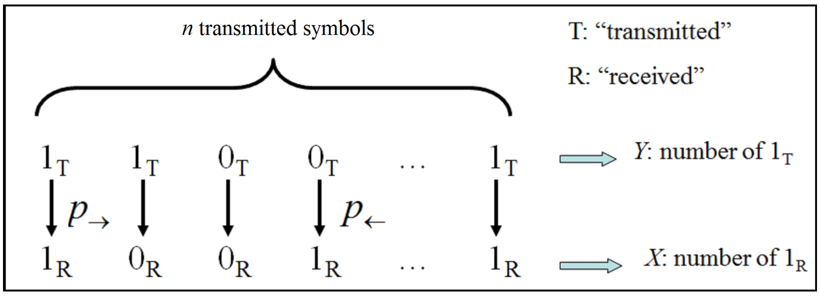

Figure 1: Counting scheme for a sequence of n transmitted symbols

3. The Bayesian model

According to Bayes-Laplace rule, the posterior probability that the measurand Y is equal to a particular value y, given that the number X of counted items is equal to x, is

$$P(Y = y|X = x) = \frac{P(X = x|Y = y)P(Y = y)}{P(X = x)} , \tag{1}$$

where $P(X = x|Y = y)$ is the likelihood function of y, $P(Y = y)$ is the prior pmf for Y and $P(X = x)$ is given by the law of total probability $P(X = x) = \sum_{y=0}^{n} P(X = x|Y = y)P(Y = y)$. The novel result of the present work is the development of the following likelihood function:

$$P(X = x|Y = y) = \sum_{t=\max\{0,\, x-y\}}^{\min\{n-y,\, x\}} \binom{y}{x-t}\binom{n-y}{t} p_{\rightarrow}^{x-t}\,(1-p_{\rightarrow})^{t-x+y}\, p_{\leftarrow}^{t}\,(1-p_{\leftarrow})^{n-y-t} , \tag{2}$$

where the symbol $\binom{\bullet}{\bullet}$ denotes the binomial coefficient. If viewed as a function of x for a fixed value y, this is the pmf of the conditional random variable $X|y$. Expression (2) is derived as the sum of two independent binomial distributions. In the framework described by Fig. 1, given a value y for the measurand, the variable $X|y$ modelling the total number of counted items is indeed the sum of $X_{\rightarrow} \sim \mathsf{Binom}(y, p_{\rightarrow})$, giving the counted existing items, and $X_{\leftarrow} \sim \mathsf{Binom}(n-y, p_{\leftarrow})$, giving the non-existing items which are also counted. It is interesting to study the behaviour of expression (2) for particular limiting values of the parameters $p_{\rightarrow}$ and $p_{\leftarrow}$. For example, when $p_{\rightarrow} = 1$ and $p_{\leftarrow} = 0$, (best case, no counting error), the pmf of $X|y$ is a Kronecker delta function centred in y. When $p_{\leftarrow} = 0$, (no double counting, i.e. no false positives), $X|y$ reduces to a binomial distribution $X|y \sim \mathsf{Binom}(y, p_{\rightarrow})$, with $E(X|y) \leq y$ for any y, that is to say that x tends to underestimate the measurand.

The prior pmf of Y conveys the information about the measurand available before counting. It is related to the total number Y of 1_T within the sequence of n symbols to be transmitted. When nothing is known about the measurand, all the y values are *a priori* equally probable, and the appropriate pmf for Y is a uniform discrete distribution on $[0, n]$, $Y \sim \mathsf{Unif}(0, n)$. In the case that $p_{\uparrow}$ is known, we use as prior a $\mathsf{Binom}(n, p_{\uparrow})$, counting the total number of 1_T within the sequence of n symbols to be transmitted.

The posterior pmf for Y, i.e. the conditional distribution of $Y|x$, is obtained from expression (1). This pmf, which constitutes the measurement result as defined in [5], can be studied for the above-mentioned prior distributions and for various values of $p_{\uparrow}$ (in the prior) and $p_{\rightarrow}$ and $p_{\leftarrow}$ (in the likelihood). Considering the two aforementioned cases, for example, in the no-counting-error situation, also the posterior density of $Y|x$ is a Kronecker delta function: it is centred at the measurement result x, irrespectively of the prior knowledge on the measurand. In the no-double-counting case, for both priors the posteriors reflect the information conveyed by the likelihood about the impossibility for the true number of items to be smaller than that actually counted, though showing a different shape because of the different prior knowledge.

4. Example 1 - Monitoring pill intake

A crucial aspect of successful medical treatment is the patient's adherence to prescribed medications. Yet, it is known that the adherence rate in patients with chronic diseases is low and approximately 50 % of patients in developed countries do not take their medications as prescribed. A tool to monitor pill intake that can be implemented in mobile health solutions was recently proposed [6] to detect the presence/absence of pills in a blister through digital images of the blister itself. The counting mode intends to detect and count remaining valid pills in a (partially) used blister. Authors estimated that for transparent blisters, for example, the rate of correct present pills detection is 88.2 %, i.e., $p_{\rightarrow} = 0.882$, whereas the rate of correct taken pills detection is 80.8 %, i.e., $p_{\leftarrow} = 1 - 0.808 = 0.192$. The tool, though, is not able to provide an uncertainty associated with an estimate of remaining pills in the blister.

Applying the model proposed in Section 3, with a Uniform prior pmf for the actual number of present pills in a blister with n pill pockets (no information on the actual status of the blister), leads to the results in Tab. 2 (for $n = 10$). Mean, mode and standard deviation of the posterior pmf are reported, the standard deviation being taken as the standard uncertainty associated with the measurand estimate (taken as the mean or the mode of the distribution). When the observed value of present pills is close to 0 or n, the corresponding standard deviation is small. It increases for intermediate x values, indicating a less informative posterior. Future incorporation of information about the treatment plan could be modelled by a different, more appropriate prior pmf, hence leading to more informative results.

5. Example 2 - Conformity assessment of a sample of items

JCGM 106:2012 [7] provides a Bayesian framework for the conformity assessment of the true value[1] η of a property of interest of a single item by inclusion of prior knowledge on η, modelled by a pdf $g_0(\eta)$, and the knowledge acquired in the measurement (leading to a

[1] "The item is distinguished by a single scalar quantity (a measurable property) defined to a level of detail sufficient to be reasonably represented by an essentially unique true value" [7].

Table 1: Parameters of variable $Y|x$ for a prior $Y \sim \mathsf{Unif}(0, 10)$, $p_{\rightarrow} = 0.882$ and $p_{\leftarrow} = 0.192$.

x	$E(Y\|x)$	Mode$(Y\|x)$	std$(Y\|x)$
0	0.17	0	0.45
1	0.57	0	0.76
2	1.17	1	1.08
3	2.01	2	1.39
4	3.11	3	1.62
5	4.41	5	1.73
6	5.79	6	1.72
7	5.79	6	1.72
8	8.27	8	1.31
9	9.14	10	0.96
10	9.72	10	0.60

measured value η_{m}, affected by measurement uncertainty and hence typically different from η). Global consumer risk R_{c} is defined as the joint probability of η being outside a tolerance interval TI and its corresponding measured value η_{m} being inside an acceptance interval AI (false positive error). The other way round is for the global producer risk R_{p} (false negative error). True positive and true negative probabilities are here indicated as p_{TP} and p_{TN}, respectively.

The JCGM 106:2012 framework considers a single item. Should a lot of n items undergo conformity assessment, the question would be "given that x items in the sample are accepted as conforming, which is the probability to actually have any other number y of truly conforming items in the whole sample"?

Model in Section 3 can be adapted to answer that question. The probability of each item to be truly conforming being $p_{\uparrow} = P(\eta \in TI) = \int_{TI} g_0(\eta) d\eta$, and $p_{\rightarrow}$ and $p_{\leftarrow}$ being, respectively, $P(\eta_{\mathrm{m}} \in AI | \eta \in TI)$ and $P(\eta_{\mathrm{m}} \in AI | \eta \notin TI)$, yields

$$p_{\rightarrow} = P(\eta_{\mathrm{m}} \in AI \cap \eta \in TI)/P(\eta \in TI) = p_{\mathrm{TP}}/p_{\uparrow} \ , \tag{3}$$

$$p_{\leftarrow} = P(\eta_{\mathrm{m}} \in AI \cap \eta \notin TI)/P(\eta \notin TI) = R_c/(1 - p_{\uparrow}) \ . \tag{4}$$

Let us consider the case $R_c = 0.006$, $R_p = 0.017$, $p_{\mathrm{TP}} = 0.970$, $p_{\mathrm{TN}} = 0.007$ (note that their sum is 1, since probabilities of mutually exclusive events). In this case, according to Eqs. (3) and (4), $p_{\rightarrow} = 0.98$ and $p_{\leftarrow} = 0.6$. For $n = 10$ tested items, $p_{\uparrow} = 0.99$ and $x = 9$ of accepted items in the sample, the posterior pmf for the items truly conforming in the sample has a mode equal to 8 and a mean equal to 6.6 (both smaller than the counted value x), with a non-negligible standard deviation of 2.4. Indeed, the probability of having less than 9 items truly conforming is a quite impressive 76 %. This is due to the high value of $p_{\leftarrow} = 0.6$ leading to a considerable overestimation of the number of actual conforming items.

6. Conclusions

The proposed model for counting accommodates any number of under and over countings and naturally takes into account prior knowledge on the measurand Y. It can be generalized to hierarchical models including possible prior knowledge on the involved parameters (n, $p_{\uparrow}$, $p_{\leftarrow}$, $p_{\rightarrow}$). It could in principle be applied to classification problems where $p_{\leftarrow}$ and $p_{\rightarrow}$ are related to probabilities of true/false positives and negatives. However, it is thought for cases in which the number n of checks is well defined, hence requiring adaptation for situations in which n is not finite or not clearly determinable.

References

[1] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, Guide to the Expression of Uncertainty in Measurement - GUM 1995 with minor corrections, JCGM 100:2008.

[2] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP, and OIML, Evaluation of measurement data - Supplement 1 to the “Guide to the expression of uncertainty in measurement” - Propagation of distributions using a Monte Carlo method, JCGM 101:2008.

[3] Bich W and Pennecchi F. Uncertainty in measurements by counting. *Metrologia*, 49(1):15–19, 2012.

[4] Shannon C E. A Mathematical Theory of Communication. *Bell System Technical Journal*, 27(3):379–423, 1948.

[5] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, International Vocabulary of Metrology - Basic and general concepts and associated terms (VIM), JCGM 200:2008.

[6] Holtkötter J, Amaral R, Almeida R, Jácome C, Cardoso R, Pereira A, Pereira M, Chon K H, and Fonseca J A. Development and Validation of a Digital Image Processing-Based Pill Detection Tool for an Oral Medication Self-Monitoring System. *Sensors*, 22:2958, 2022.

[7] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP and OIML, Evaluation of measurement data - The role of measurement uncertainty in conformity assessment, JCGM 106:2012.