# A Capacitance Build-Up Method to Determine LCR Meter Errors and Capacitance Transfer

# A capacitance build-up method to determine LCR meter errors and capacitance transfer

Ngoc Thanh Mai Tran and Vincenzo D'Elia and Luca Callegaro and Massimo Ortolano

***Abstract*—We present a capacitance build-up method suitable for the determination of the measurement error of a capacitance meter. The method requires only a small number of uncalibrated base capacitors, to be connected in parallel in various combinations, and a single calibrated capacitor, which provides measurement traceability. The outcome of the method is both the determination of the meter error and the calibration of all the base capacitors: it can therefore be considered also a capacitance scaling method. The method's equations, cast in matrix form, express estimates and uncertainties for all the quantities of interest. As an example of application, a commercial $LCR$ meter is calibrated in the ranges (100–1000) pF and (1–10) nF at 1.6 kHz and 10 kHz, with an accuracy at the level of a few parts in $10^6$. The calibration is validated by comparison with a ultra-high accuracy capacitance bridge.**

***Index Terms*—Metrology, impedance measurement, capacitance measurement, measurement uncertainty, calibration**

## I. Introduction

A numerical quantity value is the expression of the ratio of the quantity of interest to its measurement unit [1]. The measuring instruments for extensive electrical quantities, such as voltage and capacitance, perform the measurement with a ratio device (e.g., a divider) and one or more reference standards, often embedded in the instrument itself.

The calibration of electrical capacitance meters involves the measurement, using the meter under calibration, of several artifact reference standards having calibrated known values. Often, only a limited number of calibrated standards are available, typically with decadal nominal values; hence, in a given measurement range, just one or two calibration points can be probed.

Several methods have been proposed to overcome this limitation and synthesize, starting from a limited number of calibrated standards, a larger set of calibration values in the same meter range. Among them, we mention the use of inductive voltage or current dividers [2], [3] and of digital electronic synthesizers [4], [5].

In the following we present a calibration method based on capacitance build-up. The method involves, using the meter to be calibrated, the measurement of a single calibrated capacitor, and of a sequence of capacitance values generated by combining in parallel a small group of uncalibrated base capacitors. The outcome of the method is both the determination of the meter error on a large number of calibration points in the range, and the calibration of all base capacitors. It can be therefore considered also as a scaling capacitance method.

Build-up methods have a long history in metrology: for instance, in mass metrology, they are employed to generate subdivisions of the kilogram [6]. In electrical metrology, build-up methods are employed to calibrate transformer ratios [7], the voltage ratio of transformer bridges [8] and to determine the AC-DC difference of thermal converter groups [9], and also for capacitance [10]

Our method, shortly introduced in [11] in a preliminary non-general form, is here described in full with additional measurement results, together with an evaluation of the uncertainty. A variation has been published by other authors [12].

Section II presents the complete mathematical formulation of the build-up method in a general matrix form, together with the evaluation of the uncertainty. Section III presents the set-up and the results of an experiment comparing capacitance measurements obtained with an $LCR$ meter and the build-up method to those obtained with a calibrated high-accuracy capacitance meter. Finally, section IV discusses the main limitations of the method.

## II. Mathematical formulation

Consider a capacitance meter yielding the reading $C^{\mathrm{read}}(C)$ when measuring a capacitance $C$. The meter measurement error is $\ell(C) = C^{\mathrm{read}}(C) - C$. We present here a formulation of the build-up method with two goals in mind:

1) To determine $\ell(C)$ at $L$ capacitance values by generating $N$ parallel combinations of $M < L$ uncalibrated capacitors and one calibrated reference capacitor. And, as a by-product, to determine also calibration values for the uncalibrated capacitors and for the combinations.
2) To determine the ratios, corrected for the meter errors, of the capacitances of $M$ uncalibrated capacitors to that of a reference capacitor.

In both cases, we shall base the formulation on two assumptions:

A1) The equivalent capacitance of parallel capacitors is the sum of the individual capacitances. The validity of this assumption, which is apparently trivial, actually depends on the realization of the impedance definition in the experimental setup (see section IV-A).

A2) The meter error is the same for two sufficiently close capacitance values, that is, $\ell(C_1) \approx \ell(C_2)$ when $C_1 \approx C_2$. In particular, we shall assume that the meter error is the same when two capacitors have the same nominal value. This assumption is also at the basis of some substitution

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measurements and is essentially equivalent to neglecting the differential nonlinearity of the meter. Its validity can be somewhat confirmed *a posteriori* from the estimated error curve and from the analysis of the residuals of the solution (see section IV-B).

### A. Determination of meter errors and capacitances

Let us start with $M+1$ base capacitors: the first $M$ capacitors are uncalibrated with nominal values $C_{0i}^{\mathrm{nom}}$, $i=1,\ldots,M$, possibly repeated, and unknown values $C_{0i}$; the last capacitor is a calibrated reference capacitor with nominal value $C_{\mathrm{ref}}^{\mathrm{nom}}$ and known value $C_{\mathrm{ref}}$.

The $M+1$ base capacitors can be connected in parallel to generate $N$ combinations with nominal capacitances $C_k^{\mathrm{nom}}$, $k=1,\ldots,N$, possibly unknown values $C_k$ and, by measuring each combination with the capacitance meter, associated meter readings $C_k^{\mathrm{read}}$. Not all possible combinations need to be considered, and combinations consisting of a single capacitor can be included.

The measured deviation of the $k$th reading from the corresponding nominal value is

$$\delta_k^{\mathrm{read}} = C_k^{\mathrm{read}} - C_k^{\mathrm{nom}} = C_k + \ell(C_k) - C_k^{\mathrm{nom}}, \tag{1}$$

where $\ell(C_k) = C_k^{\mathrm{read}}(C) - C_k$ is the meter measurement error for the $k$th combination. From assumption A1, the nominal capacitances of the combinations and their actual values can be written as

$$C_k^{\mathrm{nom}} = \sum_{i=1}^{M} a_{ki} C_{0i}^{\mathrm{nom}} + b_k C_{\mathrm{ref}}^{\mathrm{nom}}, \tag{2}$$

$$C_k = \sum_{i=1}^{M} a_{ki} C_{0i} + b_k C_{\mathrm{ref}}, \tag{3}$$

with $a_{ki}=1$ (respectively, $b_k=1$) if the $i$th base capacitor (respectively, the reference base capacitor) participates to the $k$th combination, and 0 otherwise. Combining equations (1)–(3) yields

$$\delta_k^{\mathrm{read}} = \sum_{i=1}^{M} a_{ki}\delta_{0i} + b_k\delta_{\mathrm{ref}} + \ell(C_k), \tag{4}$$

with $\delta_{0i} = C_{0i} - C_{0i}^{\mathrm{nom}}$ and $\delta_{\mathrm{ref}} = C_{\mathrm{ref}} - C_{\mathrm{ref}}^{\mathrm{nom}}$. The latter quantity is known so that $b_k\delta_{\mathrm{ref}}$ can be moved to the left hand side, thus yielding

$$\delta_k^{\mathrm{read}} - b_k\delta_{\mathrm{ref}} = \sum_{i=1}^{M} a_{ki}\delta_{0i} + \ell(C_k). \tag{5}$$

From assumption A2, $\ell(C_k) = \ell(C_k^{\mathrm{nom}})$, and since there can be only $L \le N$ different nominal capacitance values, we can rewrite (5) as

$$\delta_k^{\mathrm{read}} - b_k\delta_{\mathrm{ref}} = \sum_{i=1}^{M} a_{ki}\delta_{0i} + \sum_{j=M+1}^{M+L} a_{kj}\ell_j, \tag{6}$$

where $\ell_j$, $j=1,\ldots,L$, is the value of the meter error corresponding to the $j$th unique nominal capacitance value, and $a_{kj}=1$ if the $k$th combination has the $j$th unique nominal capacitance value, and 0 otherwise.

The second summation in the right hand side of (6) actually consists of just one non-zero term, but its introduction allows us to put the system of equations (6) into the matrix form

$$\boldsymbol{\delta}^{\mathrm{read}} - \delta_{\mathrm{ref}}\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} \tag{7}$$

or

$$\boldsymbol{P}\boldsymbol{\delta} = \boldsymbol{A}\boldsymbol{x}, \tag{8}$$

where $\boldsymbol{\delta}^{\mathrm{read}} = [\delta_1^{\mathrm{read}},\ldots,\delta_N^{\mathrm{read}}]^\top$ ($\top$ denotes matrix transposition) is the column vector of the measured deviations, $\boldsymbol{b} = [b_1,\ldots,b_N]^\top$, $\boldsymbol{P} = [\boldsymbol{I}_N \ -\boldsymbol{b}]$, $\boldsymbol{I}_N$ is the $N\times N$ identity matrix,

$$\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\delta}^{\mathrm{read}} \\ \delta_{\mathrm{ref}} \end{bmatrix}, \tag{9}$$

$$\boldsymbol{A} = [\boldsymbol{A}_1 \ \boldsymbol{A}_2] \tag{10}$$

$$= \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1M} & a_{1(M+1)} & \cdots & a_{1(M+L)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NM} & a_{N(M+1)} & \cdots & a_{N(M+L)} \end{array}\right] \tag{11}$$

is an $N\times(M+L)$ design matrix divided into an $N\times M$ submatrix $\boldsymbol{A}_1$ and an $N\times L$ submatrix $\boldsymbol{A}_2$, and

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\delta}_0 \\ \boldsymbol{\ell} \end{bmatrix} = \begin{bmatrix} \delta_{01} \\ \vdots \\ \delta_{0M} \\ \hline \ell_1 \\ \vdots \\ \ell_L \end{bmatrix} \tag{12}$$

is the column vector of the unknown explanatory variables.

If the matrix $\boldsymbol{A}$ has full column rank, that is, it has $M+L$ linearly independent columns, the system of equations (8) has a unique solution in the least-square sense [13, chapter 14]. From a very general point of view, for an arbitrary set of numerical quantity values of the combinations (e.g. with incommensurable ratios), it may not be possible to find a suitable set of base capacitances for which $\boldsymbol{A}$ has full column rank. In practice, by restricting the set of numerical quantity values of the combinations to simple rational values, the required condition on the rank of $\boldsymbol{A}$ can be easily fulfilled by a suitable choice of the base capacitances.

The system of equations (8) can be solved either as an *ordinary least square* (OLS) problem or as a *generalized (weighted) least square* (GLS) problem [14], [15]. Here we perform the analysis for the OLS problem.

The OLS solution of (8) can be formally written as

$$\hat{\boldsymbol{x}} = \boldsymbol{A}^{+}\boldsymbol{P}\boldsymbol{\delta}, \tag{13}$$

where

$$\hat{\boldsymbol{x}} = \begin{bmatrix} \hat{\boldsymbol{\delta}}_0 \\ \hat{\boldsymbol{\ell}} \end{bmatrix} = \begin{bmatrix} \hat{\delta}_{01} \\ \vdots \\ \hat{\delta}_{0M} \\ \hline \hat{\ell}_1 \\ \vdots \\ \hat{\ell}_L \end{bmatrix} \tag{14}$$

is the estimated vector of the explanatory variables and $\boldsymbol{A}^+$ is the Moore-Penrose pseudoinverse [13], [14], [16]. When $\boldsymbol{A}$ has full-column rank, $\boldsymbol{A}^+ = (\boldsymbol{A}^\top \boldsymbol{A})^{-1}\boldsymbol{A}^\top$ [13], but we shall not use here this explicit expression because there are more numerically stable standard algorithms for the calculation of $\boldsymbol{A}^+$, and the solution of (8), that are implemented in common numerical analysis computer programs (see e.g. [14, chapter 6]).

The estimated values of the base capacitors are

$$\hat{C}_{0i} = C_{0i}^{\mathrm{nom}} + \hat{\delta}_{0i}, \tag{15}$$

which can be put in matrix form as

$$\hat{\boldsymbol{C}}_0 = \boldsymbol{C}_0^{\mathrm{nom}} + \hat{\boldsymbol{\delta}}_0, \tag{16}$$

with $\hat{\boldsymbol{C}}_0 = [\hat{C}_{01}, \ldots, \hat{C}_{0M}]^\top$ and $\boldsymbol{C}_0^{\mathrm{nom}} = [C_{01}^{\mathrm{nom}}, \ldots, C_{0M}^{\mathrm{nom}}]^\top$. Moreover, from (3), the estimated capacitances of the combinations are

$$\hat{C}_k = \sum_{i=1}^{M} a_{ki}\hat{C}_{0i} + b_k C_{\mathrm{ref}} \tag{17}$$

$$= \sum_{i=1}^{M} a_{ki}(C_{0i}^{\mathrm{nom}} + \hat{\delta}_{0i}) + b_k(C_{\mathrm{ref}}^{\mathrm{nom}} + \delta_{\mathrm{ref}}). \tag{18}$$

The latter set of equations can be written as

$$\hat{\boldsymbol{C}} = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{b}\end{bmatrix}\begin{bmatrix}\boldsymbol{C}_0^{\mathrm{nom}} + \hat{\boldsymbol{\delta}}_0 \\ C_{\mathrm{ref}}^{\mathrm{nom}} + \delta_{\mathrm{ref}}\end{bmatrix}, \tag{19}$$

where $\hat{\boldsymbol{C}} = [\hat{C}_1, \ldots, \hat{C}_N]^\top$. Finally, the $\hat{\ell}_j$'s are the estimated meter errors at $L$ different capacitance values.

### B. Uncertainty of meter errors and base capacitances

The uncertainty of the estimate $\hat{\boldsymbol{x}}$ can be obtained by substituting into (13) a suitable probability model for $\boldsymbol{\delta}^{\mathrm{read}}$ and $\delta_{\mathrm{ref}}$.

Here we assume that $\boldsymbol{\delta}^{\mathrm{read}}$ is a random vector with covariance matrix $\hat{\sigma}^2\boldsymbol{I}_N$, where

$$\hat{\sigma}^2 = \frac{\boldsymbol{r}^\top\boldsymbol{r}}{N-(M+L)} \tag{20}$$

is the sample variance estimated from the vector

$$\boldsymbol{r} = \boldsymbol{P}\boldsymbol{\delta} - \boldsymbol{A}\hat{\boldsymbol{x}} \tag{21}$$

of the residuals of the solution (13) [15]. We also assume that $\delta_{\mathrm{ref}}$ is a random variable with standard deviation equal to the associated uncertainty $u(\delta_{\mathrm{ref}}) = u(C_{\mathrm{ref}})$ of the reference base capacitor and that $\boldsymbol{\delta}^{\mathrm{read}}$ and $\delta_{\mathrm{ref}}$ are uncorrelated. With these assumptions, the covariance matrix of $\boldsymbol{\delta}$ is

$$\operatorname{cov}\boldsymbol{\delta} = \begin{bmatrix}\hat{\sigma}^2\boldsymbol{I}_N & \boldsymbol{0}_{N\times 1} \\ \boldsymbol{0}_{1\times N} & u^2(\delta_{\mathrm{ref}})\end{bmatrix}. \tag{22}$$

From (13) and elementary properties of covariance matrices[1], we can write the covariance matrix of $\hat{\boldsymbol{x}}$ as

$$\operatorname{cov}\hat{\boldsymbol{x}} = \boldsymbol{A}^+\boldsymbol{P}(\operatorname{cov}\boldsymbol{\delta})\boldsymbol{P}^\top(\boldsymbol{A}^+)^\top. \tag{23}$$

[1]We shall use here and in subsequent derivations, without further notice, the following properties of covariance matrices. Given random vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$, constant vectors and matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{A}$ and $\boldsymbol{B}$, it holds that: $\operatorname{cov}(\boldsymbol{y},\boldsymbol{x}) = \operatorname{cov}(\boldsymbol{x},\boldsymbol{y})^\top$, $\operatorname{cov}(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z}) = \operatorname{cov}(\boldsymbol{x},\boldsymbol{z}) + \operatorname{cov}(\boldsymbol{y},\boldsymbol{z})$ and $\operatorname{cov}(\boldsymbol{a}+\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}+\boldsymbol{B}\boldsymbol{y}) = \boldsymbol{A}\operatorname{cov}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{B}^\top$ (see e.g. [17, Theorem 5.7] with a straightforward generalization to random vectors).

Substituting the expression of $\boldsymbol{P}$ and that of $\operatorname{cov}\boldsymbol{\delta}$ into the above equation yields

$$\operatorname{cov}\hat{\boldsymbol{x}} = \boldsymbol{A}^+(\boldsymbol{A}^+)^\top\hat{\sigma}^2 + \boldsymbol{A}^+\boldsymbol{b}\boldsymbol{b}^\top(\boldsymbol{A}^+)^\top u^2(\delta_{\mathrm{ref}}). \tag{24}$$

The first term of (24) represents the type A uncertainty component of the measurement, whereas the second term the type B component due to the uncertainty of the reference base capacitor.

To evaluate the uncertainties of the estimates $\hat{\boldsymbol{C}}_0$, from (16), and $\hat{\boldsymbol{C}}$, from (19), the covariance matrix $\operatorname{cov}\hat{\boldsymbol{x}}$ can be suitably decomposed in the four covariance submatrices

$$\operatorname{cov}\hat{\boldsymbol{x}} = \begin{bmatrix}\operatorname{cov}\hat{\boldsymbol{\delta}}_0 & \operatorname{cov}(\hat{\boldsymbol{\delta}}_0, \hat{\boldsymbol{\ell}}) \\ \operatorname{cov}(\hat{\boldsymbol{\ell}}, \hat{\boldsymbol{\delta}}_0) & \operatorname{cov}\hat{\boldsymbol{\ell}}\end{bmatrix}. \tag{25}$$

Furthermore, from (13) and (14), the covariance between $\hat{\boldsymbol{x}}$ and $\delta_{\mathrm{ref}}$ is

$$\operatorname{cov}(\hat{\boldsymbol{x}}, \delta_{\mathrm{ref}}) = \begin{bmatrix}\operatorname{cov}(\hat{\boldsymbol{\delta}}_0, \delta_{\mathrm{ref}}) \\ \operatorname{cov}(\hat{\boldsymbol{\ell}}, \delta_{\mathrm{ref}})\end{bmatrix} = -\boldsymbol{A}^+\boldsymbol{b}u^2(\delta_{\mathrm{ref}}). \tag{26}$$

From (16),

$$\operatorname{cov}\hat{\boldsymbol{C}}_0 = \operatorname{cov}\hat{\boldsymbol{\delta}}_0 \tag{27}$$

and, finally, from (19),

$$\operatorname{cov}\hat{\boldsymbol{C}} = \boldsymbol{A}_1\operatorname{cov}\hat{\boldsymbol{\delta}}_0\boldsymbol{A}_1^\top + \boldsymbol{A}_1\operatorname{cov}(\hat{\boldsymbol{\delta}}_0, \delta_{\mathrm{ref}})\boldsymbol{b}^\top + \boldsymbol{b}\operatorname{cov}(\delta_{\mathrm{ref}}, \hat{\boldsymbol{\delta}}_0)\boldsymbol{A}_1^\top + \boldsymbol{b}\boldsymbol{b}^\top u^2(\delta_{\mathrm{ref}}). \tag{28}$$

The diagonal elements of the above calculated covariance matrices yield the uncertainties of the estimates of interest,

$$u(\hat{C}_{0i}) = \sqrt{(\operatorname{cov}\hat{\boldsymbol{C}}_0)_{ii}} \tag{29}$$

and

$$u(\hat{C}_k) = \sqrt{(\operatorname{cov}\hat{\boldsymbol{C}})_{kk}}. \tag{30}$$

### C. Determination of capacitance ratios

We want to determine here the ratios $\hat{w}_{0i}$ between the estimated base capacitances $\hat{C}_{0i}$ and the reference capacitance $C_{\mathrm{ref}}$. Letting $\hat{\boldsymbol{w}}_0 = [\hat{w}_{01}, \ldots, \hat{w}_{0M}]^\top$, from (16), we can write

$$\hat{\boldsymbol{w}}_0 = \frac{1}{C_{\mathrm{ref}}}\hat{\boldsymbol{C}}_0 \tag{31}$$

$$= \frac{1}{C_{\mathrm{ref}}^{\mathrm{nom}} + \delta_{\mathrm{ref}}}(\boldsymbol{C}_0^{\mathrm{nom}} + \hat{\boldsymbol{\delta}}_0) = \hat{\boldsymbol{w}}_0\left(\begin{bmatrix}\hat{\boldsymbol{\delta}}_0 \\ \delta_{\mathrm{ref}}\end{bmatrix}\right), \tag{32}$$

that is, we can consider $\hat{\boldsymbol{w}}_0$ as a function of $\hat{\boldsymbol{\delta}}_0$ and $\delta_{\mathrm{ref}}$.

### D. Uncertainty of capacitance ratios

Following [18], we can write the covariance matrix of (32) as

$$\operatorname{cov}\hat{\boldsymbol{w}}_0 = \boldsymbol{J}_{\hat{\boldsymbol{w}}_0}\operatorname{cov}\left(\begin{bmatrix}\hat{\boldsymbol{\delta}}_0 \\ \delta_{\mathrm{ref}}\end{bmatrix}\right)\boldsymbol{J}_{\hat{\boldsymbol{w}}_0}, \tag{33}$$

where

$$\boldsymbol{J}_{\hat{\boldsymbol{w}}_0} = \frac{1}{C_{\mathrm{ref}}^{\mathrm{nom}} + \delta_{\mathrm{ref}}}\begin{bmatrix}\boldsymbol{I}_M & -\hat{\boldsymbol{w}}_0\end{bmatrix} \tag{34}$$

is the Jacobian (sensitivity) matrix of the function (32) with respect to the input quantities $\hat{\boldsymbol{\delta}}_0$ and $\delta_{\mathrm{ref}}$, and

$$\operatorname{cov}\left(\begin{bmatrix}\hat{\boldsymbol{\delta}}_0 \\ \delta_{\mathrm{ref}}\end{bmatrix}\right) = \begin{bmatrix}\operatorname{cov}\hat{\boldsymbol{\delta}}_0 & \operatorname{cov}(\hat{\boldsymbol{\delta}}_0, \delta_{\mathrm{ref}}) \\ \operatorname{cov}(\delta_{\mathrm{ref}}, \hat{\boldsymbol{\delta}}_0) & u^2(\delta_{\mathrm{ref}})\end{bmatrix}. \tag{35}$$

In the above covariance matrix, the elements $\operatorname{cov}\hat{\boldsymbol{\delta}}_0$ and $\operatorname{cov}(\hat{\boldsymbol{\delta}}_0, \delta_{\mathrm{ref}})$ can be obtained, respectively, from (25) and (26).

TABLE I: Capacitance ranges and $LCR$ meter ranges used in the experiment.

| Capacitance range | Frequency | $LCR$ meter range | Applied voltage |
|---|---|---|---|
| (100–1000) pF | 1.6 kHz | 30 kΩ | 1 V |
| (100–1000) pF | 10 kHz | 10 kΩ | 1 V |
| (1–10) nF | 1.6 kHz | 10 kΩ | 100 mV |
| (1–10) nF | 10 kHz | 1 kΩ | 100 mV |

## III. Experimental set-up and results

The build-up method is here applied to determine the error of an $LCR$ meter (Agilent 4284A), the values of the uncalibrated base capacitors from a single calibrated reference capacitor, and the ratios between the base capacitors and the reference capacitor. The results are then compared with those obtained from a high accuracy capacitance meter (Andeen-Hagerling AH2700A) calibrated against the Italian national capacitance scale. The measurements are taken at 1.6 kHz and 10 kHz across the capacitance ranges (100–1000) pF and (1–10) nF. For each capacitance range and frequency, the $LCR$ meter range was held fixed, according to the values reported in table I, and the automatic level control was activated. An open-short calibration was performed before each set of measurements.

### A. Capacitors set-up and combinations

For both ranges, there are six base capacitors of which $M = 5$ are uncalibrated and one is calibrated against the Italian national capacitance scale with a relative uncertainty of $10^{-7}$. Table II lists the base capacitors, specifying for each capacitor the nominal value, manufacturer and dielectric material. The base capacitance values are adjusted close to their nominal values by the addition of small parallel correction capacitors.

All capacitors are individually shielded to reduce cross capacitances and temperature controlled at 23 °C. The uncalibrated capacitors are kept in a thermostatic chamber (Kambič TK-190 US) with a temperature stability better than 4 mK [19]; the reference capacitor is individually temperature controlled [20].

The capacitors are defined as two-terminal pair impedances, in which the connecting cables are part of the definition. Two junction boxes are used to parallel the capacitors. The two-terminal pair definition is referenced at the junction boxes.

Table III shows, as example, the list of the capacitance combinations for the (1–10) nF range. A "1" indicates that a capacitor is included in a combination, whereas a "0" means that a capacitor is left unconnected. In this list, all possible combinations are included. For example, the combination with $k = 6$, with a nominal value of 3 nF, includes $C_{03} = 2\,\mathrm{nF}$ and $C_{\mathrm{ref}} = 1\,\mathrm{nF}$. This example combination is represented in figure 1 together with the connections to the $LCR$ meter.

A photograph of the base capacitors is shown in figure 2.

### B. Results

*1) Estimation of meter error and capacitance:* We here determine the error of the $LCR$ meter and the values of the uncalibrated base capacitors across the capacitance ranges (100–1000) pF and (1–10) nF, at 1.6 kHz and 10 kHz.

The relative deviations

$$\epsilon(\hat{C}_{0i}) = \frac{\hat{C}_{0i} - C_{0i}^{\mathrm{AH2700}}}{C_{0i}^{\mathrm{nom}}} \tag{36}$$

of the base capacitance values $\hat{C}_{0i}$ estimated by the build-up method from the reference values $C_{0i}^{\mathrm{AH2700}}$ obtained with the high-accuracy capacitance meter are shown in figure 3. The orange error bars represent the relative uncertainties $u(\hat{C}_{0i})/C_{0i}^{\mathrm{nom}}$, as estimated from (29). The blue error bars represent the combined uncertainties $u(\epsilon(\hat{C}_{0i}))$ which also account for the uncertainty of the Andeen-Hagerling AH2700, as reported by the specifications [21]. All the uncertainties are represented with a coverage factor $k = 1$. The resulting uncertainties of the build-up method are at the level of a few parts in $10^6$, and the relative deviations $\epsilon(\hat{C}_{0i})$ are generally compatible with zero.

Figure 4 shows an example of the residuals $r_k$ for each combination, as calculated from (21), at 1.6 kHz and in the (1–10) nF range.

Figure 5 shows the estimated meter errors $\hat{\ell}_j$ together with the associated uncertainties. The plots show that the meter error is dominated by the gain error.

*2) Capacitance ratio:* We here determine the ratio of the estimated base capacitances to the reference capacitance, again across the capacitance ranges (100–1000) pF and (1–10) nF, at 1.6 kHz and 10 kHz.

Figure 6 shows the relative deviations

$$\epsilon(\hat{w}_{0i}) = \frac{\hat{w}_{0i} - w_{0i}^{\mathrm{AH2700}}}{w_{0i}^{\mathrm{nom}}} \tag{37}$$

in which $\hat{w}_{0i}$ is the $i$th capacitance ratio estimated from (32), $w_{0i}^{\mathrm{AH2700}}$ is the capacitance ratio measured by the high-accuracy capacitance meter and $w_{0i}^{\mathrm{nom}}$ is the nominal capacitance ratio. Since the reference capacitance is 1000 pF, the nominal ratios in the (100–1000) pF range are 0.1, 0.1, 0.2, 0.2 and 0.5, whereas the nominal ratios in the (1–10) nF range are 1, 1, 2, 2 and 5.

The results are generally compatible. It is worth noting that in figure 6(d) the larger combined uncertainty, with respect to the other measurements, is caused by an abrupt increase of the uncertainty of the AH2700 at 10 nF.

## IV. Limitations

The build-up method presented in the previous sections is based on assumptions A1 and A2. If these assumptions do not hold, the results are affected by additional errors that should be possibly prevented or taken into account with an appropriate modelling.

### A. Limitations on A1

Assumption A1 may fail due to an imperfect impedance definition, and this can happen in two ways. First, if the impedance measurement is not referenced at the junction boxes, the voltage at the low terminal pairs of the capacitors is no longer zero and depends on the number of connected

TABLE II: List of the base capacitors employed in the experiment.

| Label | $C^{\mathrm{nom}}$ | Manufacturer | Model | Description |
|---|---|---|---|---|
| | | | | (100–1000) pF range |
| $C_{01}$ | 100 pF | Vishay | VP32BA101FC | C0G, ceramic capacitor |
| $C_{02}$ | 100 pF | Vishay | VP32BA101FC | C0G, ceramic capacitor |
| $C_{03}$ | 200 pF | Vishay | VP32BA101FC (2×) | C0G, ceramic capacitor |
| $C_{04}$ | 200 pF | Vishay | VP32BA101FC (2×) | C0G, ceramic capacitor |
| $C_{05}$ | 500 pF | Vishay | VP32BA470FC | C0G, ceramic capacitor |
| $C_{\mathrm{ref}}$ | 1 nF | General Radio | 1404-A | Gas capacitor, temperature controlled, calibrated, $u(C_{\mathrm{ref}})/C_{\mathrm{ref}} = 1 \times 10^{-7}$ |
| | | | | (1–10) nF range |
| $C_{01}$ | 10 nF | Vishay | | C0G, ceramic capacitor |
| $C_{02}$ | 1 nF | General Radio | 1404-A | Gas capacitor |
| $C_{03}$ | 2 nF | General Radio | 1409-F | Mica capacitor |
| $C_{04}$ | 2 nF | General Radio | 1409-F | Mica capacitor |
| $C_{05}$ | 5 nF | General Radio | 1409-K | Mica capacitor |
| $C_{\mathrm{ref}}$ | 1 nF | General Radio | 1404-A | Gas capacitor, temperature controlled, calibrated, $u(C_{\mathrm{ref}})/C_{\mathrm{ref}} = 1 \times 10^{-7}$ |

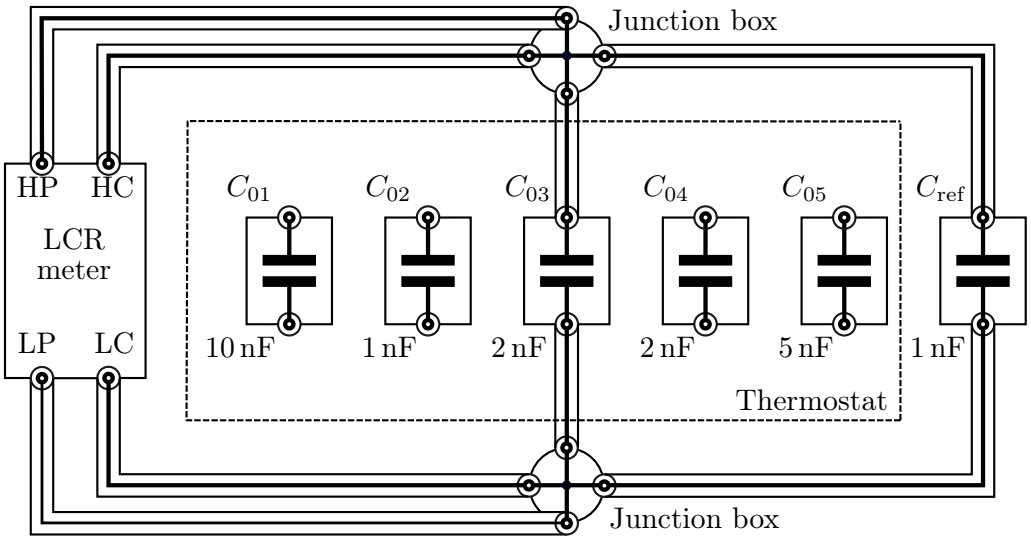



Fig. 1: Schematic of the build-up method with the base capacitors for the capacitance range from 1 nF to 10 nF. As an example, the base capacitors are connected to form the 6th combination of table III.

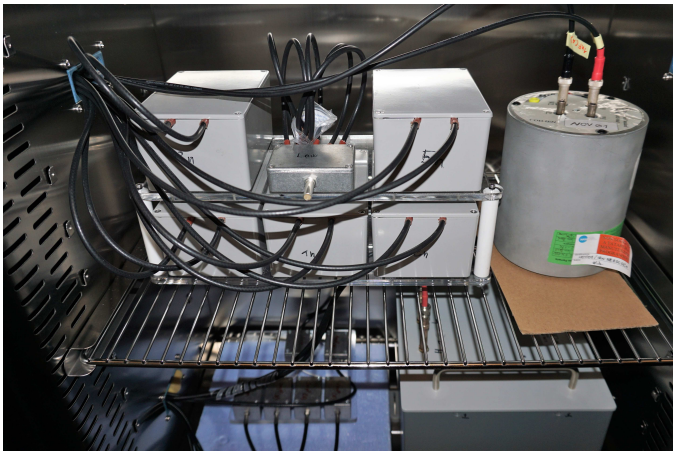


Fig. 2: Photograph of the base capacitors used in the experiment in the range from 1 nF to 10 nF. The capacitors are individually shielded and placed inside a thermostatic chamber.

capacitors. When necessary, this effect can be corrected with circuit modelling. Second, the coaxiality of a system with many different parallel combinations cannot be easily ensured with the usage of coaxial equalizers because their number and placement would depend on the combination. In this sense, a compact construction may be preferred (see e.g. [12], [22]–[25]). Furthermore, this method is easily applicable only to the case of capacitors defined as two-terminal-pair impedances. For capacitors defined as four-terminal-pair impedances it would be necessary to implement compensation networks for each capacitor [26].

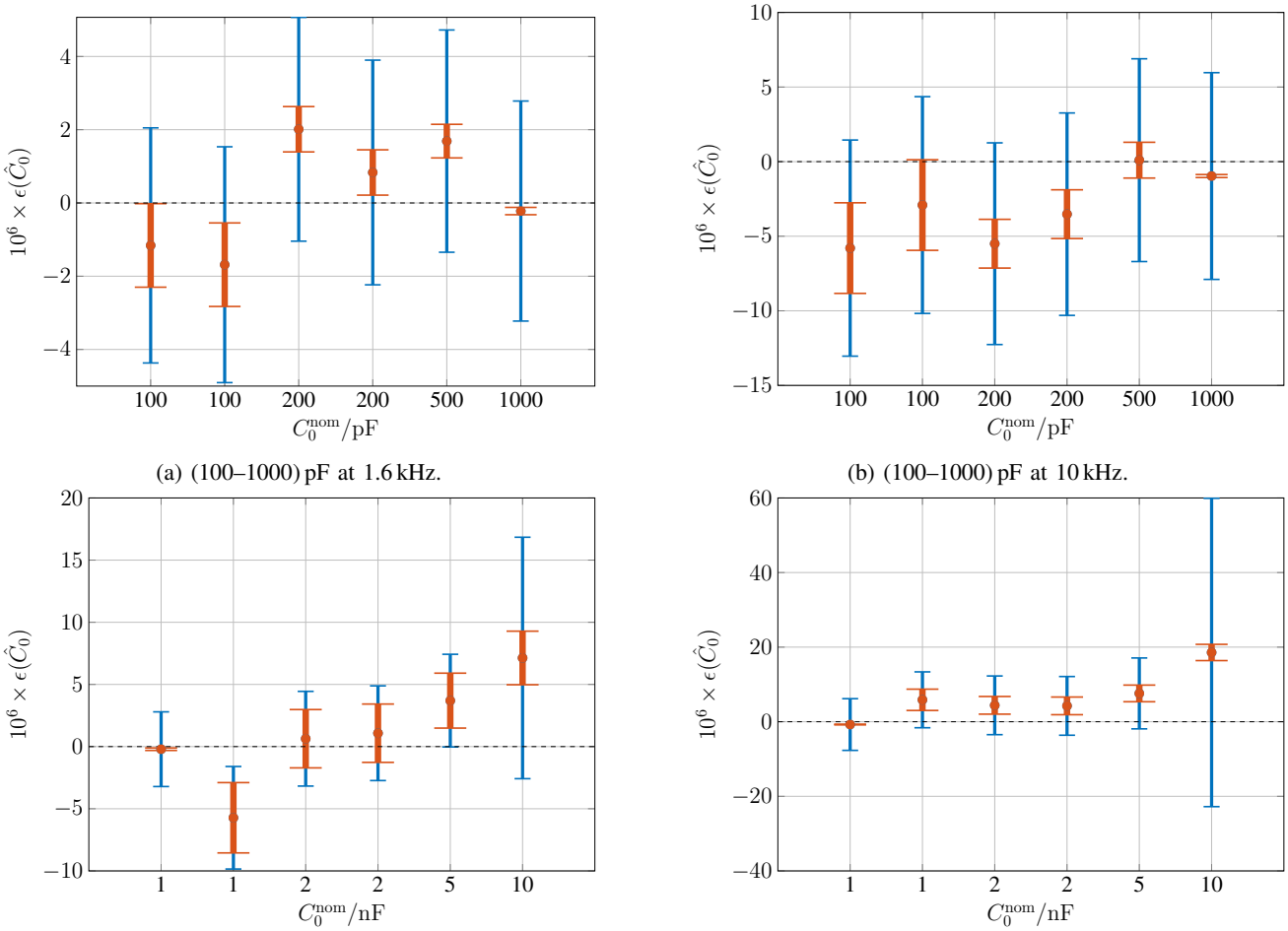
### B. Limitations on A2

To analyze the limitations of assumption A2, let us develop $\ell(C_k)$ in a first-order Taylor expansion around $\ell(C_k^{\mathrm{nom}})$,

$$\ell(C_k) \approx \ell(C_k^{\mathrm{nom}}) + \frac{\mathrm{d}\ell}{\mathrm{d}C}(C_k^{\mathrm{nom}})(C_k - C_k^{\mathrm{nom}}), \tag{38}$$

$$\approx \ell(C_k^{\mathrm{nom}}) + g(C_k^{\mathrm{nom}})\delta_k^{\mathrm{read}}, \tag{39}$$

where $g(C_k^{\mathrm{nom}}) = (\mathrm{d}\ell/\,\mathrm{d}C)(C_k^{\mathrm{nom}})$ represents the differential nonlinearity of the error curve around $C_k^{\mathrm{nom}}$. Therefore, from




(a) (100–1000) pF at 1.6 kHz.

(b) (100–1000) pF at 10 kHz.

(c) (1–10) nF at 1.6 kHz.

(d) (1–10) nF at 10 kHz.

Fig. 3: The relative deviation $\epsilon(\hat{C}_0)$ between the estimated values of the base capacitances from the build-up method and the high-accuracy capacitance meter. The thick orange error bars represent the uncertainties of the build-up method, whereas the combined uncertainty is represented by the thin blue error bars. The reference capacitance is the 1000 pF capacitance represented last in (a) and (b), and first in (c) and (d).

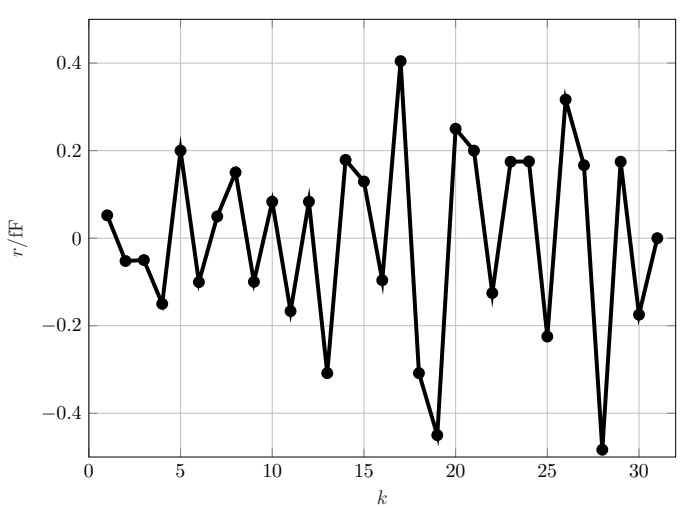
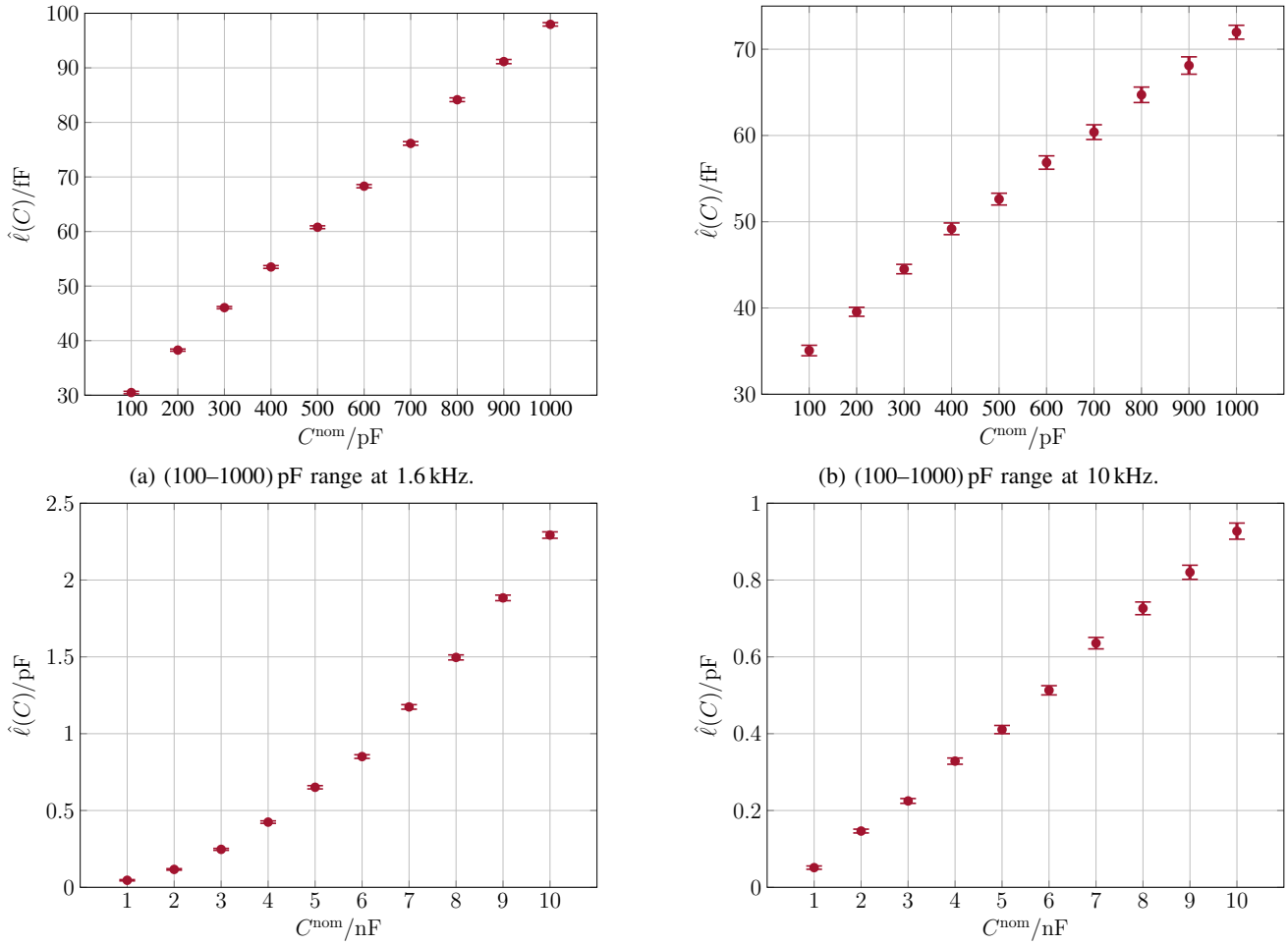



Fig. 4: The residuals $r_k$ of the build-up solution in the (100–1000) pF range at 1.6 kHz

a practical point of view, assumption A2 can be considered met when $|g(C_k^{\mathrm{nom}})\delta_k^{\mathrm{read}}| \ll u(\hat{C}_k)$, that is, when the effect of the differential nonlinearity is sufficiently less than the evaluated uncertainty. This condition can be achieved by trimming the base capacitors sufficiently close to their nominal values (section III-A) to reduce the magnitude of $\delta_k^{\mathrm{read}}$.

For example, from the results of figure 5 (rough estimates), $g \approx 8 \times 10^{-5}$ for the (100–1000) pF range at 1.6 kHz, $g \approx 4 \times 10^{-5}$ for the (100–1000) pF range at 10 kHz, $g < 4 \times 10^{-4}$ for the (1–10) nF range at 1.6 kHz, and $g \approx 1 \times 10^{-4}$ for the (1–10) nF range at 10 kHz. From the raw data, we have that the maximum of the relative deviations $|\delta_k^{\mathrm{read}}|/C_k$ is about $3 \times 10^{-3}$ for the (100–1000) pF range and about $3 \times 10^{-4}$ for the (1–10) nF range. The relative effect of the differential nonlinearity for these measurements can be thus estimated in the $10^{-8}$–$10^{-7}$ range, maximum a few parts in $10^7$, and therefore considered negligible with respect to the evaluated uncertainty.

Assumption A2 may fail in particular if the $LCR$ meter changes gain or offset along the measurement range, making




(a) (100–1000) pF range at 1.6 kHz.

(b) (100–1000) pF range at 10 kHz.

(c) (1–10) nF range at 1.6 kHz.

(d) (1–10) nF range at 10 kHz.

Fig. 5: The meter error $\ell(C)$ at $L$ capacitance valued with the associated uncertainties for each range and frequency of the experiment.

the transfer characteristic discontinuous around the combination values. For this reason, the $LCR$ meter range should be held fixed along the whole measurement range. Possibly abrupt variations of the local gain may be detectable from the analysis of the residuals (21).

## V. Conclusions

The capacitance build-up method here presented allows to perform meter calibrations and capacitance scaling. The presented application examples showed that the method accuracy is much better than the specified accuracy of a top-class $LCR$ meter and comparable with that of a high-accuracy capacitance bridge. The method is implementable with commercial instrumentation and just one calibrated standard, and it is therefore suitable for industrial calibration centers. An implementation embedded into an instrument might also be considered to allow the instrument self-calibration.

The method can be possibly extended to other quantities and instruments, such as, for example, high DC resistances measured by a two-terminal high-accuracy ohmmeter, by directly replacing the capacitance with the conductance in all equations.

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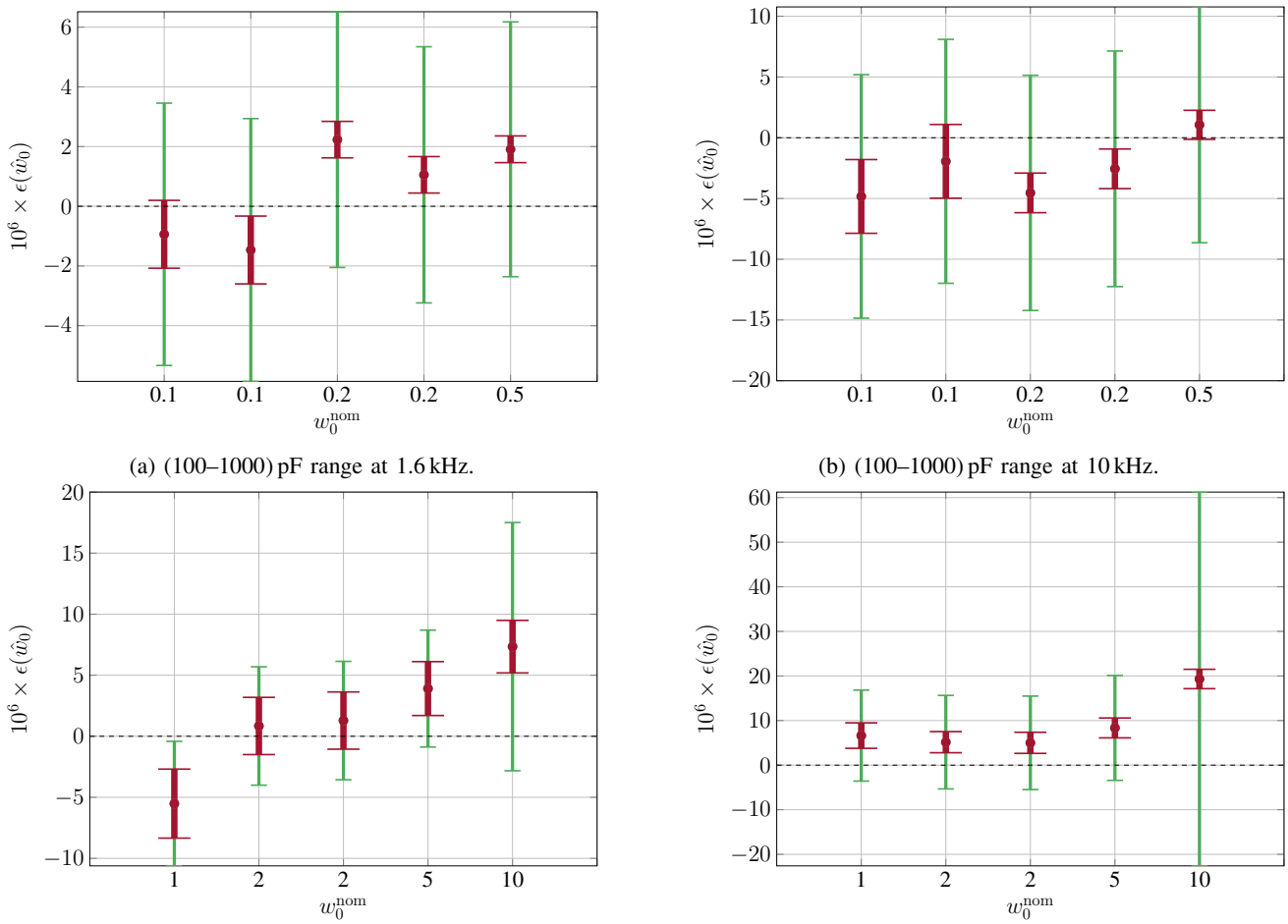


(a) (100–1000) pF range at 1.6 kHz.

(b) (100–1000) pF range at 10 kHz.

(c) (1–10) nF range at 1.6 kHz.

(d) (1–10) nF range at 10 kHz.

Fig. 6: The relative deviation $\epsilon(\hat{w}_{0i})$ between the estimated values of the capacitance ratios from the build-up method and those from the high-accuracy capacitance meter. The thick red error bars represent the uncertainties of the build-up method, whereas the combined uncertainty is represented by the thin green error bars.


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TABLE III: List of the capacitance combinations for the (1–10) nF range. A "1" indicates that a capacitor is included in a combination, whereas a "0" means that a capacitor is left unconnected. These values define the elements of the matrix $\boldsymbol{A}_1$ defined in (11).

| $k$ | $C_{01}^{\mathrm{nom}}$ 10 nF | $C_{02}^{\mathrm{nom}}$ 1 nF | $C_{03}^{\mathrm{nom}}$ 2 nF | $C_{04}^{\mathrm{nom}}$ 2 nF | $C_{05}^{\mathrm{nom}}$ 5 nF | $C_{\mathrm{ref}}^{\mathrm{nom}}$ 1 nF | $C_k^{\mathrm{nom}}$ |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 nF |
| 2 | 0 | 1 | 0 | 0 | 0 | 0 | 1 nF |
| 3 | 0 | 0 | 1 | 0 | 0 | 0 | 2 nF |
| 4 | 0 | 0 | 0 | 1 | 0 | 0 | 2 nF |
| 5 | 0 | 1 | 0 | 0 | 0 | 1 | 2 nF |
| 6 | 0 | 0 | 1 | 0 | 0 | 1 | 3 nF |
| 7 | 0 | 0 | 0 | 1 | 0 | 1 | 3 nF |
| 8 | 0 | 1 | 1 | 0 | 0 | 0 | 3 nF |
| 9 | 0 | 1 | 0 | 1 | 0 | 0 | 3 nF |
| 10 | 0 | 0 | 1 | 1 | 0 | 0 | 4 nF |
| 11 | 0 | 1 | 1 | 0 | 0 | 1 | 4 nF |
| 12 | 0 | 1 | 0 | 1 | 0 | 1 | 4 nF |
| 13 | 0 | 0 | 0 | 0 | 1 | 0 | 5 nF |
| 14 | 0 | 0 | 1 | 1 | 0 | 1 | 5 nF |
| 15 | 0 | 1 | 1 | 1 | 0 | 0 | 5 nF |
| 16 | 0 | 0 | 0 | 0 | 1 | 1 | 6 nF |
| 17 | 0 | 1 | 0 | 0 | 1 | 0 | 6 nF |
| 18 | 0 | 1 | 1 | 1 | 0 | 1 | 6 nF |
| 19 | 0 | 0 | 1 | 0 | 1 | 0 | 7 nF |
| 20 | 0 | 0 | 0 | 1 | 1 | 0 | 7 nF |
| 21 | 0 | 1 | 0 | 0 | 1 | 1 | 7 nF |
| 22 | 0 | 0 | 1 | 0 | 1 | 1 | 8 nF |
| 23 | 0 | 0 | 0 | 1 | 1 | 1 | 8 nF |
| 24 | 0 | 1 | 1 | 0 | 1 | 0 | 8 nF |
| 25 | 0 | 1 | 0 | 1 | 1 | 0 | 8 nF |
| 26 | 0 | 0 | 1 | 1 | 1 | 0 | 9 nF |
| 27 | 0 | 1 | 1 | 0 | 1 | 1 | 9 nF |
| 28 | 0 | 1 | 0 | 1 | 1 | 1 | 9 nF |
| 29 | 0 | 0 | 1 | 1 | 1 | 1 | 10 nF |
| 30 | 0 | 1 | 1 | 1 | 1 | 0 | 10 nF |
| 31 | 1 | 0 | 0 | 0 | 0 | 0 | 10 nF |

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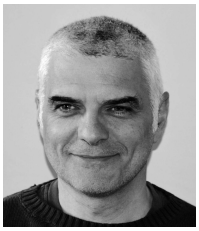

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