A device for measuring the Moment of Inertia for aerospace applications

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***Abstract*—The knowledge of the Moment of Inertia of space components and small satellites is of paramount importance for the attitude control of the same. A simple device based on a precision air bearing equipped with a torsion rod has been realized at INRIM. The principle, the practical realization and the calibration procedure are presented. The apparatus has been tested with objects having mass less than 5 kg and moment of inertial up to 0.034 kg m^2 obtaining an the accuracy is of the order of 0,2 %.**

Keywords—Moment of Inertia, torsion pendulum

I. Introduction

The inertial properties of a body consist of the mass, the center of gravity, the MOI (Moment of Inertia) and the products of inertia. The moment of inertia is one of the main parameters needed to model the dynamic behavior of aircraft and spacecraft in aerospace engineering. The mass can be measured easily, it is a rather difficult task to determine the CoG (Centre of Gravity), but is much more difficult the measure of the MOI.

When the MOI of the aircraft is not accurately measured, the direction will be very difficult to adjust, which can lead to serious problems for the control, with a consequent increase of the consumption. Improving the measurement precision of the MOI has become an inevitable requirement in aerospace industry.

Calculation methods based on three-dimensional solid model based on the CAD model and FEM pre-processor have a low precision, which usually does not meet the requirements, so that a direct measurement is required.

There are different methods to determine the inertial moment, which employ experiments based on the physical principle of the pendulum, that is on the relation between the harmonic oscillations and the MOI of the body under test.

A method usually used for large bodies is realized by a platform suspended by bi-filar, tri-filar or multi-filar pendulum [1], on which the body under test is placed. The relative accuracy of all these approaches varies from 0.3 % to 1%.

A different approach usually applied for small object is based on the torsion pendulum where the oscillation of the object placed on rotating table is obtained by a torsion rod [2]. With this method the accuracy can be at the order of 0.1% to 0.2%.

The apparatus recently developed at INRIM is based on the torsion pendulum principle, and has been built with the scope of measuring the MOIs Baffles of the cameras of the Telescope Optical Units (TOU) of the PLAnetary Transits and Oscillations of stars (PLATO), which is the third medium-class mission in ESA's Cosmic Vision program [3]. Its objective is to find and study a large number of extrasolar planetary systems, with emphasis on the properties of terrestrial planets in the habitable zone around solar-like stars. The design of the TOU is in charge to Thales Alenia Space. The pictures of two out of the three different Baffles to be characterized are given in Fig. 1.

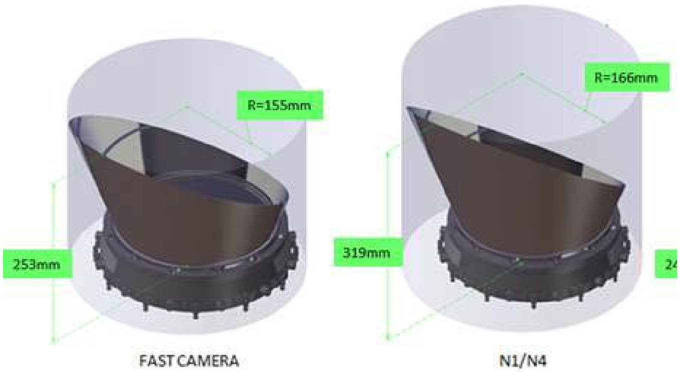

Fig. 1. Two out of the trhee kind of baffles to be used on the PLATO mission cameras.

II. Measurement model

Differently from the physical pendulum, in a torsion pendulum the energy is stored in a torsion spring or rod. In the ideal case of undamped oscillation and in absence of external forces, after an initial angular displacement with respect to the rest position of the spring, the system will oscillate at its natural frequency, depending on the mass moment of inertia and the torsional stiffness of the spring.

The homogeneous equation of motion for the torsion pendulum, under the assumption of undamped oscillation can be written as:

$$\ddot{\vartheta} + \omega_n^2 \vartheta = 0 \quad (1)$$

where ϑ is the angular position and ω_n the angular frequency, which is referred to as natural frequency and is defined as:

$$\omega_n = \sqrt{\frac{k}{I}} \quad (2)$$

where k is the torsional stiffness of the spring and I is the MOI of the rotating mass. From Equation (2) it is possible to obtain the measurement model for the MOI, which can be written as:

$$I = \frac{k}{(2\pi)^2} T^2 \quad (3)$$

where T is the time period of oscillation.

In the case of a not negligible viscous damping, the homogeneous equation of motion of the torsion pendulum becomes:

$$\ddot{\vartheta} + \mu\dot{\vartheta} + \omega_n^2 \vartheta = 0 \quad (4)$$

where $\mu = \beta/I$, where β is the viscous coefficient. If $\omega_n > \mu/2$, after an initial angular displacement with respect to the rest position of the spring, the system will oscillate with an exponentially decreasing amplitude, at an angular frequency ω_s lower than the natural frequency ω_n, which is given by:

$$\omega_s = \sqrt{\omega_n^2 - \frac{\mu^2}{4}} \quad (5)$$

From Equation (5), it is possible to obtain a relation which allows to express the time period of oscillation as function of the MOI:

$$T^{-2} = \frac{k}{(2\pi)^2} I^{-1} - \left(\frac{\beta}{4\pi}\right)^2 I^{-2} \quad (6)$$

Equations (3) and (6) can be used as measurement models for the MOI in the case of undamped and damped oscillations respectively. In both cases, the torsional stiffness of the spring and the viscous coefficient that characterize the measurement system should be determined by calibration. In particular, the calibration constants of the torsion pendulum are obtained by the regression analysis of the measurement data of oscillation period corresponding to objects with known MOI values.

III. MEASUREMENT SYSTEM

The apparatus has been realized with a rotating system in which the device under test in placed on a precision rotary table (diameter of 203 mm) supported by an air bearing system. The restoring force is given by a spring mounted under the bearing fixed to the optical table. The measuring system is in fact a device aimed to determine the natural oscillation period. A laser beam is mounted externally to the table, and a photodetector is placed at a distance of about 500 mm. A mirror is fixed to the table rotating with the same. The laser beam impinges on the mirror that deflects the same towards the photodetector when the table is at the rest position. When the table is moved from its rest position and is left free to oscillate, the mirror acts as an optical lever making the beam swing horizontally. For each oscillation period the beam hits the photo detector twice (corresponding to the zero positions) generating fast rising and falling signals. These signals are sent to an ADC board and analyzed by means of a LabView based software, the output of which is the duration of each period calculated by summing the two half periods. A picture of the system is given in Fig. 2.

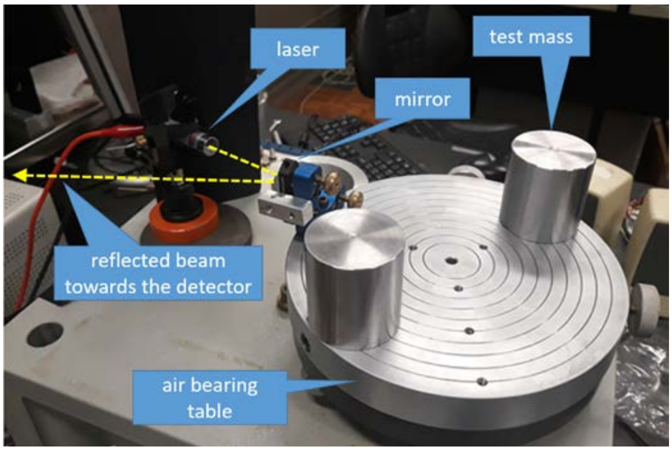

Fig. 2. Picture of the apparatus.

The oscillation of the measuring system if forced manually, twisting the spring of 90°, in this way the number of oscillations are between 30 and 32, and are independent from the weight of the object on the table.

A typical time period measurement is given in Fig. 3.

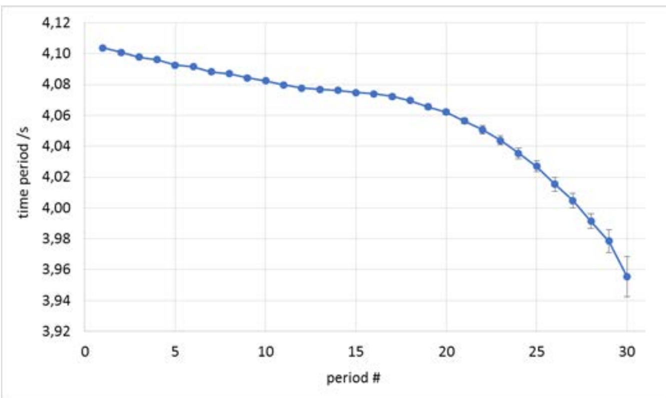

Fig. 3. Period value measured for each oscillation of the pendulum

From Fig. 3 a non-linear behavior of the adopted measurement system can be observed, as the measured value of the oscillation period depends on the number of counted periods, that is on the oscillation amplitude. This can be likely due to the type and mounting of the torsion spring, which entail a slightly different dynamic of the system as function of the torsion angle.

In order to improve the accuracy, several measurements of the time period of oscillation are taken for an interval of oscillation amplitude, in which the absolute value of the relative deviation of the period is lower than 0.05%, as shown in Fig. 4.

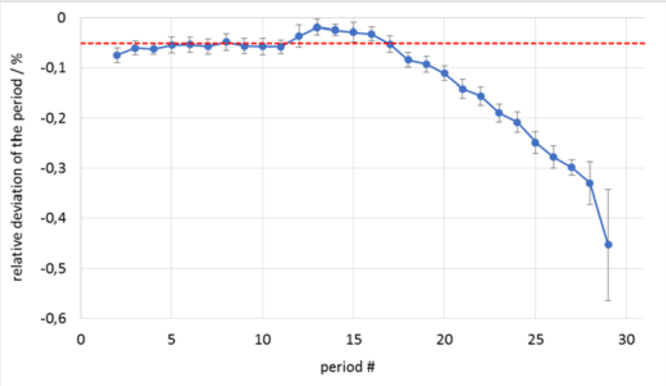

Fig. 4. Relative deviation of the period as function of the number of oscillations

It has been also checked the reproducibility of the measurements, repeating the measurements in different days, it is about of 0.1% , which probably depends on the temperature. For this reason, in order to increase the accuracy it is suggested to perform the calibration of the apparatus before the measurement.

This method allows measuring the MOI with respect to an arbitrary axis of the object under test, which is defined by the position of the object with respect the rotation axis of the table.

When the object has the geometrical center coincident with the center of the table, the MOI with respect the axis passing through the CoG I_{CG} will be

$$I_{CG} = I - m\,r^2 \tag{7}$$

where m is the mass and I is the MOI with respect to the axis passing through the geometrical center and r the displacement between the geometrical centre of the object and the CoG.

IV. Calibration

Considering the measurement model described by Equation (3), the torsional stiffness of the measuring system, that is, more generally, the calibration constant of the instrument, can be determined by calibration. The simplified equation to measure the MOI I of an object under test by a torsion pendulum can be expressed as:

$$I = C(T^2 - T_0^2) \tag{8}$$

where C is a calibration constant for the instrument, which is related to the torsional stiffness of the measuring system. As the period T_0 is the time period of the measurement system without the object under test, it can be also considered as a constant parameter (which depends on the test fixture mounted on the rotating table).

Thus, after having determined the calibration parameters C and T_0, the moment of inertia of object under test can be calculated by measuring of the oscillation period T of the object under test placed on the apparatus.

The calibration of the system was performed by measuring the oscillation period of different objects (cylinders and tubes), whose MOI were determined by their shapes and positions on the table. In Table 1 mass and dimensions of these objects are given.

Object (qty)	Outer radius / m	Inner radius / m	Mass / kg
Cylinder (2)	0.034 35		2.061 85
Tube 1 (1)	0.068 07	0.050 29	2.703 10
Tube 2 (1)	0.083 82	0.077 46	5.203 44

Table 1: objects used for the test, made of steel.

The uncertainties associated to the radius are $u(r)$ = 0.01 mm for the cylinders and $u(r)$ = 0.025 mm for the tubes, whereas for the mass was $u(m)$ = 0.020 g

The objects were placed vertically on the table with the CoG on the centre of the table; in addition, the two cylinders were placed in opposite position at 180° on the edge of the table, at a distance R=67.35 mm from the center (with $u(R)$ = 0.05 mm). For this last configuration the MOI is given by:

$$I = 2\,I_c + 2\,m\,R^2 \tag{9}$$

where I_c and m are the MOI and the mass of the two cylinders.

In Table 2 the different configurations and the related MOI with the associated uncertainties are given.

Object	MOI / kg m^2	u(MOI) / kg m^2	u(MOI) /MOI
Cylinder	0.001 216 4	0.000 000 7	0.06 %
Tube 1	0.009 681 3	0.000 004 6	0.05 %
Two cylinders at R	0.021 138	0.000 028	0.13 %
Tube 2	0.033 893 8	0.000 011 9	0.04 %

Table 2: Calculated moment of inertia of the test objects and the associated uncertainty.

The uncertainty of the MOI standards I_s, also depends on the positioning of CoG of the standards in the centre of the table: an additional contribution should be taken into account, which has been evaluated considering that the measured MOI is:

$$I_s = I + m\,\Delta_{\mathrm{CoG}}^2 \tag{10}$$

As the uncertainty associated to Δ_{CoG} was evaluated as $u(\Delta_{CoG})$ = 0.03 mm, this additional contribution, being always less than 5×10^{-9} kg m^2 is negligible.

The repeatability S of the oscillation period has been evaluated taking several measurements at approximately the same oscillation amplitude.

According to Equation (8), the fitting curve of the MOI values I *vs* the differences $T^2 - T_0^2$ is linear. As the variances associate to the x and y variables are not constant, the fitting parameters were estimated by the weighted Total Least Square method [4], where the covariance matrix associated with $T^2 - T_0^2$ is a diagonal matrix with the variances $u^2(T^2 - T_0^2) = (2T)^2u^2(T) + (2T_0)^2u^2(T_0)$ on its diagonal, where the uncertainties associate to the periods are evaluate form the standard deviation of the mean value, that this $u(T) = S/\sqrt{n}$ where n is the number of readings. The covariance matrix associated with I is a diagonal matrix with the variance $u^2(I)$.

Averaging several readings further improves accuracy: in our test 10 readings were taken.

The results are given in Table 3.

Object	Period / s	S / s
Empty	3.717 05	0.003 0
Cylinder	3.763 78	0.000 59
Tube 1	4.073 36	0.000 65
Two cylinders at R	4.456 81	0.001 45
Tube 2	4.851 36	0.000 89

Table 3: Experimental results obtained by the period measurement.

In Fig. 5 an example of the linear regression of experimental data is shown.

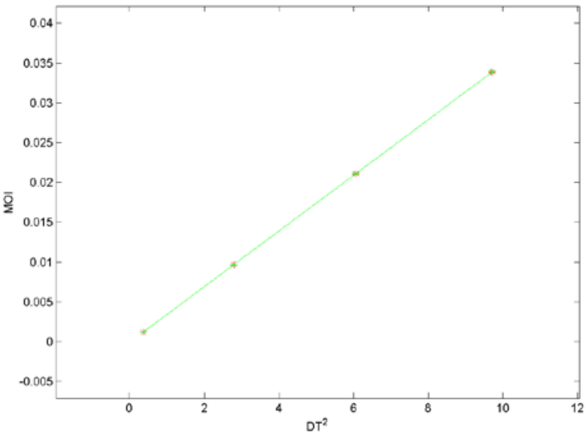

Fig. 5. Linear regression by Equation (8) for various experimental data.

Calibration Parameter	Estimate	Standard uncertainty
T_0 / s	3.717 05	0.000 94
C / kg m^{2}s^{-2}	0.003 488 7	0.000 002 6

Table 4: Experimental results obtained by regression analysis of period measurement.

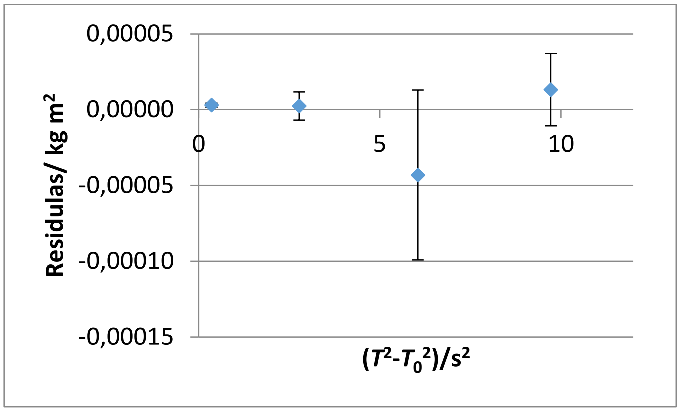

Fig. 6. Residuals of the fit as in Fig. 5.

The χ^2 has consistent values, compared with the degrees of freedom ν =n-p, χ^2/(n-p) = 0.46.

In Fig. 6 the graph of the residual is shown, where the bar are the associated expanded uncertainties to the MOI standards. Only for the first point the residual (30×10^{-7} kg m^2) is bigger than the expanded uncertainty (14×10^{-7} kg m^2), this results shows the limit of this apparatus, that is the minimum value of the MOI depends on the MOI of the rotating table.

In our case, being a table not designed for the MOI measurements the value is rather high, from Equation (8) , I = 0.048 201 kg m^2.

In addition, it is noted that the accuracy of the system is limited on the accuracy of the MOIs standards, however standard uncertainty of the order of 0.1 % are more than acceptable for this measuring system.

By using Equation (8) the uncertainty of the estimated I is evaluated as:

$$u(I) = \sqrt{(T^2 - T_0^2)^2\, u^2(C) + C^2[u(T^2) + u(T_0^2)] + u^2(K)} \quad (11)$$

An example of uncertainty budget is given in Table 5.

Quantity X_i	Estimate $\mu(X_i)$	Standard uncertainty $u(X_i)$		Standard uncertainty contribution $u_i(I)$ / kg m^2
C	0.003 488 7 kg m^{2}s^{-2}	0.000 002 6 kg m^{2}s^{-2}		$2.91 \cdot 10^{-5}$
T_0	3.717 05 s	0.000 94 s		$3.3 \cdot 10^{-6}$
T	5.000 00 s	0.000 50 s		$1.7 \cdot 10^{-6}$
I	0.039 015 5 kg m^2	Standard uncertainty	$u(I)$ / kg m^2	$2.94 \cdot 10^{-5}$
		Relative standard uncertainty	$u(I)/\mu(I)$	0.075 %

Table 5: Uncertainty budget for the MOI measurement.

With this method, the system has been calibrated with regular objects with the CoG coincident with the rotation axis of the table. However, the system was also tested in the case there is an offset between the CoG and the rotation axis. This configuration has been simulated by positioning the two cylinders at the previous distance *R* from centre, but at at 90°, thus creating a considerable unbalance of the weight on the table. In this configuration an increasing of the period of about 0.004 s, has been observed, which for the MOI corresponds to a variation of the 0.5%. This effect is due to the leveling error of the air bearing, that can be eliminated performing the measurement with the CoG of the object corresponding to the centre of the table.

V. Conclusions

With the aim to evaluate the MOI of the Baffles needed for the the ESA mission PLATO, at INRIM an apparatus has been commissioned.

The measurement is based on the torsion pendulum, which is made with a system with an air bearing and a torsion spring. A system based on a laser beam allows to measure the oscillation period accurately. After a calibration procedure based on simple geometry known masses it is possible to evaluate the MOI of any unknown object. The system has been tested for masses lower than 5 kg, but the air bearing table is capable of working with masses of tens of kilograms. Preliminary experimental results and an evaluation of uncertainty is presented. The uncertainty obtained is better than 0.2 % for MOI of the order of 0.03 kg m^2, which is better than the ESA requirements.

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