The Misconception of Closed Magnetic Flux Lines

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27 July 2024

The misconception of closed magnetic flux lines

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Abstract— The belief that magnetic flux lines are always closed has been widely disseminated, handed down from Faraday through the present day. We review the problem and revisit the example of "the wire and the loop," which shows analytically that flux lines are not necessarily closed, and extend its analysis. The pictorial representation based on flux lines may give rise to erroneous conclusions not inherent in Maxwell's laws.

Index Terms—Electromagnetics, Fundamental properties, Magnetism theory, Magnetic flux, Solenoidal fields, Helical fields, Tokamak.

I.   INTRODUCTION

"Magnetic flux lines always form closed curves." Statements of this kind are written in many textbooks and probably the reader is used to them. Unfortunately, as we shall see, they are not correct and represent a widely taught misconception about the properties of the magnetic flux density vector.

The idea of closed magnetic flux lines (also called "field lines," "streamlines," or "lines of magnetic force") is often associated to the equation that describes the divergence of the magnetic flux density $\boldsymbol{B}$

$$\nabla \cdot \boldsymbol{B} = 0 . \tag{1}$$

Moreover, this idea has been erroneously proposed as a requirement to explain the experimental fact that an isolated magnetic monopole has never been observed. The first well-documented analysis of the latter property seems to go back to the investigations performed in the thirteenth century by Pierre Pelerin de Maricourt ("Peregrinus"). Peregrinus used a natural magnet, of spherical shape, and marked over its surface the lines along which a compass needle oriented itself when put in close proximity to the magnet, obtaining meridian curves encircling it [Whittaker 1951, Elliott 1993]. This experiment, described in the *Epistola Petri Peregrini de Maricourt ad Sygerum de Foucaucourt, militem, de magnete* [Thompson 1902], gave a first awareness about the fact of inseparable magnetic poles. A long time later, in the nineteenth century, the concept of "lines of magnetic force" was largely adopted by Michael Faraday, who wrote that "every line of magnetic force is a closed curve, which in some part of its course passes through the magnet to which it belongs" [Faraday 1855]. Faraday's ideas were further developed by J. Clerk Maxwell, who provided a mathematical description for them. In his lectures *On Faraday's Lines of Force*, Maxwell stated that the lines of magnetic force produced by a closed current form closed curves embracing the current itself [Maxwell 1855]. A few years later, on the basis of such lectures, Maxwell wrote *On Physical Lines of Force* [Maxwell 1861], where (1) appeared for the first time (it can be recognized in expression (56) of this reference). The idea of closed $\boldsymbol{B}$-field lines (and, correspondingly, of closed flux tubes) seems to be deeply-rooted in the post-Maxwellian age; the interested reader can look it up for

example in a paper by J. H. Poynting [1885].

Consider now the integral form of (1)

$$\oint_{\partial \Omega} \boldsymbol{B} \cdot d\boldsymbol{s} = 0 , \tag{2}$$

where $\partial \Omega$ is a closed surface. Solenoidality is often seen as a synonymous for line "closure" [Pantazis 2017], but actually it simply requires that the flux entering $\partial \Omega$ always equals the flux outgoing from it, without the need to infer any specific global behavior of each single flux line. To satisfy (1) and (2), the presence of closed $\boldsymbol{B}$-field lines is a sufficient condition, but not strictly a necessary one. Note that, as long as one considers magnetically homogeneous domains only, these comments remain valid also for the magnetic field $\boldsymbol{H}$.

II.   PROBLEM OVERVIEW

Many textbooks support the idea that $\boldsymbol{B}$-field lines are always closed (or, at most, extend to infinity). Among them, we may mention those by Stratton [1941], Morse and Feshbach [1953], Sommerfeld [1964], Weber [1965] (who associates closed lines to any solenoidal vector), Paul and Nasar [1982], and Griffiths [1999]. The same happens in many handbooks of electrical engineering (see for instance Laughton [2003] and Schmitt [2002]). In most cases, the presence of closed $\boldsymbol{B}$-field lines is indicated as an imperative requirement to meet the property of solenoidality. Some other textbooks, like Jefimenko [1966] (where an extended use of iron filings is used to "draw" pictures of magnetic fields), speak about closed $\boldsymbol{B}$-field lines too, but associate them to a non-vanishing curl.

One of the earliest discussions about non-closed magnetic flux lines was given by the Nobel laureate I. Y. Tamm. In his book (published in Russian, since 1929, and translated in Tamm [1979]), he analyzed the field produced by a straight wire and a circular concentric loop, which will be further analyzed below. Another very interesting discussion was provided by J. Slepian [1951], who underlined that, as far as local phenomena are completely described by local vector fields, the density and direction of the lines are the only important elements, whereas the global behavior of an individual line (including its continuity) is irrelevant. Following some subtle reasoning, he also
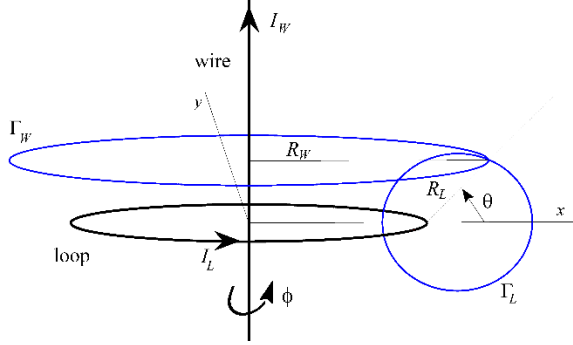
Fig. 1. Geometry for the example of the wire and the loop.

inferred that magnetic flux lines are not closed in general. A step forward was by K. L. McDonald [1954], who, with reference to the case of the wire and the loop, put in evidence the possibility of drawing field maps where the line density is infinite in a region where the field magnitude is finite. In this case, in order to restore the correspondence between the line density and the field magnitude, the lines must be artificially broken (an idea already mentioned by Slepian). McDonald investigated also the problem of singular points, i.e., points of the domain where the field magnitude vanishes and non-closed flux lines start or end. The reader can image such a situation considering a pair of ideal Helmholtz coils fed with two equal but opposite currents (so that they "push" their magnetic fields against each other). If we start from the center of each coil and follow the direction of the field, we arrive at the center of the system, where the field magnitude is zero, and we cannot go any further. Thus, these two longitudinal lines terminate there. On the other hand, given the rotational symmetry of the system, an infinite number of divergent lines perpendicular to the axis of the coils emanate radially from the singular point (which marks the "boundary" of all these lines, where *B* must necessarily vanish because it cannot have different directions at the same time). Note that singular points by no means represent magnetic monopoles or violate (1) and (2). Further examples of non-closed lines were discussed, on sound mathematical ground, by S. S. Ştefănescu [1958, 1970], who, *inter alia*, indicated Liénard [1921] as the very first reference on the subject.

Famous authors who demonstrated an awareness of non-closed magnetic lines are Rosser [1968] (who referred to Slepian), Purcell [1985] (who analyzed the case of a realistic solenoid), and Van Bladel [2007] (who mentioned both non-closed lines and singular points). A slightly different viewpoint is given in Feynman [1963], where it is written that *B*-field lines usually form closed curves, but in some complicated situations they are not simple closed loops; they do not begin or end, and never diverge from points (Feynman's description evidently does not consider the case of singular points). Renowned textbooks that, to the best of our knowledge, do not seem to support the idea of closed *B*-field lines (but neither criticize them explicitly) are Jackson [1962], Panofsky [1962] and Lorrain [1988].

In recent years, a few papers discuss the fact that the *B*-field lines do not exhibit simple closed-loops in general. Hosoda and colleagues [2009] investigated some realistic configurations, including the case of printed circuit boards. In their examples, the *B*-field lines are not closed and, in addition, the behavior can be considered chaotic (i.e., when drawing a line, a relatively small displacement of the starting point can produce a completely different trajectory). Moreover, the

perturbation introduced by the geomagnetic field sometimes generates chaos in simple non-chaotic configurations; in other cases, the effect of the terrestrial field makes the lines escape to infinity (instead of being bounded in a finite region). Thus, they came to the conclusion that non-closed lines are almost ubiquitous. In a similar analysis, Lieberherr [2010] was able to obtain a quite complex pattern of non-closed lines simply by simulating the magnetic field of a single, realistic, non-planar circuit composed of four straight segments.

III. THE WIRE AND THE LOOP REVISITED

A. Theory

Consider the situation sketched in Fig.1, including an infinitely long and straight wire placed along the *z*-axis, carrying a stationary current I_W. Due to symmetry, the generic *B*-field line associated to such a current is a closed circle Γ_W, concentric with the wire. Consider also a planar, circular and filamentary loop placed on the *xy*-plane, concentric with the *z*-axis and carrying a stationary current I_L. A generic *B*-field line Γ_L associated to the loop lies on a meridian plane (e.g. the *xz*-plane in Fig.1) and has the shape of a deformed closed circle (still planar, but "squashed" in some way). If Γ_L is rotated about the *z*-axis, we obtain a surface Σ, similar to a torus of revolution but with non-circular cross section, where the flux density produced by the loop ($\boldsymbol{B}_L$) is tangent, of course. On the other hand, the flux density produced by the wire ($\boldsymbol{B}_W$), which is azimuthal, results to be tangent to Σ in any of its points too. Thus, when both the wire and the loop are energized, the composition of $\boldsymbol{B}_L$ and $\boldsymbol{B}_W$ gives rise to a field distribution that is still tangent to Σ. More specifically, starting from a point that belongs to Σ and following the direction of the total flux density one obtains a helix, which wraps around Σ. In general, as we are going to show, such a helix does not close on itself.

Excellent analyses of the example of the wire and the loop can be found in Gascon [2005] and Aguirre [2007]. In the first paper, the authors studied some properties of the *B*-field lines (to analyze the motion of charged particles subjected to the Lorentz force and their confinement) investigating the existence of first integrals of the flux density. In the second paper, the attention was focused on possible chaotic behaviors of the flux lines, identified through perturbation techniques. To show that the helix wrapped around Σ is not closed and, therefore, dense on Σ itself, here we repropose Tamm's original approach [Tamm 1979], which is very straightforward and, unfortunately, almost neglected. First, we recall that the construction of a flux line implies that the corresponding vector field is tangent to it in any point. Thus, for a given point, the elementary displacement *dP* that must be followed to proceed along the *B*-field line is such that

$$\boldsymbol{B} \times d\boldsymbol{P} = 0 , \qquad (3)$$

which implies, in Cartesian coordinates, $dx/B_x = dy/B_y = dz/B_z$.

This condition can be easily fitted to the case of the wire and the loop because the corresponding flux densities $\boldsymbol{B}_W$ and $\boldsymbol{B}_L$ are orthogonal everywhere and hence can be seen as two components of the total field. With reference to Fig.1, $R_W d\phi$ gives an element of length for a flux line Γ_W associated to the wire, where R_W is the distance between the considered point and the wire, while ϕ is the azimuthal angle, which provides the rotation about the *z*-axis. For a flux line Γ_L associated to the loop (which is a "squashed" circle), adopting polar coordinates R_L (minimum distance between the considered point and the loop) and θ (polar angle, whose pole of

rotation is given by the intersection of the loop with the meridian plane where Γ_L lies), the element of length can be written as

$$\sqrt{\left(R_L d\theta\right)^2 + dR_L^2} = \sqrt{R_L^2 + \left(\frac{dR_L}{d\theta}\right)^2}\, d\theta \doteq R_L'd\theta .\qquad (4)$$

Thus, for a point that belongs to Σ, the flux line satisfies:

$$\frac{R_W d\phi}{B_W} = \frac{R_L'd\theta}{B_L} .\qquad (5)$$

In (5), B_W, B_L, R_W and R_L' depend on θ. Starting from a point on Σ characterized by a pair of angles ϕ_0 and θ_0, an integration gives

$$\phi - \phi_0 = \int_{\theta_0}^{\theta} \frac{B_W}{B_L} \frac{R_L'}{R_W} d\theta .\qquad (6)$$

Due to the relatively complex behavior of B_L (which could be described through elliptic integrals or harmonic expansions [Simpson 2001, Jackson 1962]), it is not easy to provide an explicit solution for (6). However, taking into account that B_W and B_L are proportional to I_W and I_L, respectively, the solution of (6) can be written as

$$\phi - \phi_0 = \frac{I_W}{I_L}\left[F(\theta) - F(\theta_0)\right],\qquad (7)$$

where function $F(\theta)$ does not depend on the values of the currents.

Some properties of $F(\theta)$ can be identified by realizing that the four quantities in the integrand of (6) are periodic functions of θ (with a 2π period). Moreover, R_W and R_L' are strictly positive quantities, whereas B_W and B_L may be positive or negative but, moving along the flux line, they do not change their sign. Thus, the integrand in (6) in turn repeats itself every 2π rad and, apart from the factor I_W/I_L, it has a non-null average value k over this period. Hence, $F(\theta)$ can be decomposed into two terms:

$$F(\theta) = k\theta + G(\theta) .\qquad (8)$$

In the right-hand side of (8), the first term accounts for the integral of the average value k, while the second term is the integral of the periodic part (with zero average value) of the normalized integrand in (6), which therefore still exhibits a periodicity of 2π. Now, a helix wrapped around Σ is closed only if an integer (positive or negative) number n_W of rotations around the wire corresponds to an integer (positive or negative) number n_L of rotations around the loop, that is

$$2\pi n_W = \frac{I_W}{I_L}\left[F(\theta_0 + 2\pi n_L) - F(\theta_0)\right].\qquad (9)$$

Taking into account (8), this implies

$$2\pi n_W = \frac{I_W}{I_L}\left[k\theta_0 + 2k\pi n_L + G(\theta_0 + 2\pi n_L) - k\theta_0 - G(\theta_0)\right],\quad (10)$$

and finally, by virtue of the periodicity of $G(\theta)$,

$$n_W I_L = k\, n_L I_W .\qquad (11)$$

It is now evident that the arbitrary choice of the starting point (which fixes the value for k), as well as that of I_W and I_L, in general does not allow satisfying (11). In this case, the helix never passes twice in the same point, but develops indefinitely and covers surface Σ completely. Thus, likewise to Peano's curve, such ergodic space-filling helix touches every point of Σ without self-intersections. Note that, if helices of this kind are used to define a flux tube of finite cross section, the tube will intersect itself when sufficiently extended in length.

We point out that the use of the (unphysical) single, infinite, wire should not be a matter of concern. Indeed, a field similar to B_W (i.e.
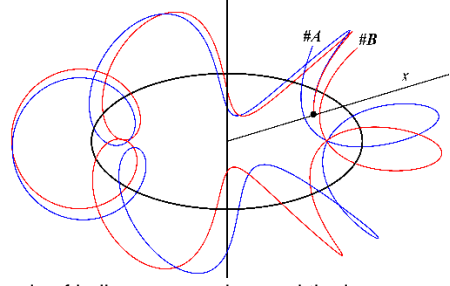

Fig. 2. Example of helices wrapped around the loop.

azimuthal, decreasing with the distance from the longitudinal z-axis) can be generated, for instance, inside a coaxial cable where the current flowing in the inner conductor returns along the shield [Paul 1982, Laughton 2003]. The same situation is obtained in the interior region of an ideal toroidal solenoid [Weber 1965, Purcell 1985, Griffiths 1999]. Thus, if this solenoid embraces a concentric loop, helical flux lines appear as well. The latter situation finds application in tokamak reactors, which, incidentally, were one of Tamm's research topics.

B. Numerical investigations

The basic procedure to draw a flux line could be the following. For a given point P_i, compute the flux density $B(P_i)$. Then, apply a "small" displacement Δ in the direction of $B(P_i)$ to find the new point P_{i+1}:

$$P_{i+1} = P_i + \Delta \frac{B(P_i)}{|B(P_i)|} .\qquad (12)$$

Unfortunately, this strategy results to be too rough because, at each step of computation, it introduces an error that puts the trajectory away from the right one (analogously to what happens using the explicit Euler method for solving a differential equation). For example, the adoption of (12) to draw the line Γ_W produced by the straight wire alone (which should be a closed circle) starting at P_{start} = (1, 0, 0) m with Δ = 1 mm, produces a spiral path that widens gradually, so that, after a complete rotation around the z-axis, the distance from the axis becomes ~1.003 m (instead of the theoretical value of 1 m). This result clearly represents a numerical artifact. An advanced (even if still simple) strategy to draw magnetic flux lines can be found in a technical report written by J. R. Pasta and S. M. Ulam in 1953 (now available in Ulam [1990]), when the authors used the early computer MANIAC to investigate heuristically the case of the wire and the loop itself (probably this was one of the very first applications of a computer to draw magnetic flux lines). Starting from P_i, such an approach uses (12) to compute a first provisional point P_i'; then, considering P_i', it applies again (12) to find a second fictitious point P_i''. The coordinates of the "true" point P_{i+1} are finally obtained as the average $(P_i + P_i'')/2$. When applying this strategy to draw one turn of the line Γ_W with P_{start} = (1, 0, 0) m and Δ = 1 mm, the deviation from the theoretical value of 1 m is bounded below $8 \cdot 10^{-10}$ m. All flux lines presented hereafter have been drawn according to this strategy, checking the stability of the results against a reduction of the step Δ.

Figure 2 shows two flux lines (#A and #B) obtained when the radius of the loop is 1 m, I_L = 1 A and the starting point is P_{start} = (0.75, 0, 0) m (this position, indicated with a thick black point, is characterized by ϕ_0 = 0 and θ_0 = π). In case of line #A, the current in the wire is I_{WA} = 1 A and the helix requires less than six (negative) complete rotations around the loop to perform one rotation around the z-axis. Line #B has been obtained with I_{WB} = 0.9 A; in this case, after six complete

rotations around the loop the helix has not completed the rotation around the z-axis yet. Thus, a suitable tuning of I_W between I_{WA} and I_{WB} allows satisfying (approximately) condition (11), with $n_W = 1$, $n_L = -6$ and $I_L = 1$ A. This has been verified empirically, setting $I_W = 0.9689$ A (in this case, a complete rotation $\phi = 2\pi$ corresponds to $|\theta - \theta_0| = 6\cdot(2\pi)$ with an error of about 2.6 mrad). By virtue of (11), this gives $|k| \approx 0.172$ for the considered P_{start}. Such a value has been also confirmed by the (independent) numerical computation of the average value of the integrand in (6), evaluated over an interval $|\theta - \theta_0| = 2\pi$.

If we keep $I_L = I_W = 1$ A and move the starting point towards the wire, e.g $P_{start} = (0.25, 0, 0)$ m, the behavior of the helix is "reversed," i.e., the helix rotates many times around the wire while turning around the loop for the first time. This situation is depicted in Fig.3a, which helps to understand that the flux line always lays on the surface obtained by rotating about the z-axis the proper curve Γ_L (the closed line passing through P_{start} when $I_L = 1$ A and $I_W = 0$ A). For an intermediate starting position, e.g. $P_{start} = (0.6, 0, 0)$ m, the helix exhibits a mixed behavior, because it "sews" itself around the wire and the loop at the same time. An example is given in Fig.3b, where it is possible to sense that, also in this case, the line keeps tangent to the surface Σ generated revolving the corresponding curve Γ_L.

IV. DISCUSSION AND CONCLUSIONS

The above results put in evidence the complexity that can be exhibited by $\boldsymbol{B}$-field lines. It must be noted that, up to now, the discussion has been restricted to stationary currents, but things become even more interesting for time-varying fields. Consider, for instance, what happens if I_L is a stationary current of 1 A, while I_W is a slowly-varying sinusoidal current with a peak of 1 A. Setting $P_{start} = (0.75, 0, 0)$ m, in correspondence to the peak of I_W the situation is that given by line #A in Fig.2, whereas, when $I_W = 0$ A, we have a line like Γ_L in Fig.1. Between these two situations (i.e., time instants), for the same P_{start}, we find an infinite number of helices with different pitch, including those that are closed. In particular, as I_W approaches zero, there is an infinite number of occurrences able to satisfy (11) for $n_W = 1$ (with increasing values of $|n_L|$). It is important to realize that, for each single time instant, in principle it is possible to draw a complete "snapshot" of the flux line, even in those cases when it extends indefinitely. In case of high-frequency fields, the discussion is analogous, but in general, depending on the geometry of the sources, a given line (entirely drawn with reference to a specific instant) may involve points where, due to the propagation, the field has different angular phases. With reference to the evolution of a switch-on transient in the time domain, at a given instant the field (which propagates at finite speed) is bounded in a finite region "behind" the front of the expanding wave. Of course, the behavior of the flux lines will be consistent with this situation if the field is computed properly, for example, applying the so-called Jefimenko's equations [Jefimenko 1966] (which actually had already appeared in Panofsky [1962]). In passing, we note that there is no unique way to create an "animation" of field lines for time-varying problems [Belcher 2003].

Another point that deserves some attention is the connection between the existence of closed lines and the curl of the corresponding vector field. In the presence of a closed line (like Γ_W in Fig.1), the circulation of the vector computed along such a line is surely different from zero. By virtue of Stokes' theorem, this means that the curl of
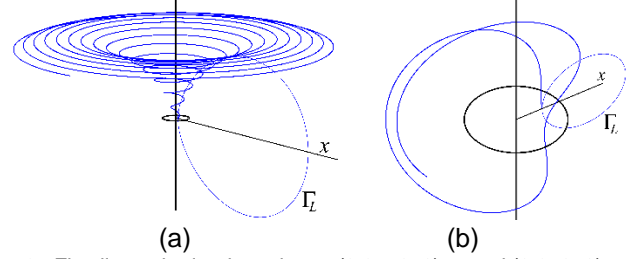

Fig. 3. Flux lines obtained starting at (0.25, 0, 0) m and (0.6, 0, 0) m.

the vector must be non-null in some points belonging to any open surface having the closed line as contour. On the contrary, the presence of a non-vanishing curl does not imply closed lines, because Stokes' theorem applies to any geometric closed path, which does not necessarily coincide with a flux line. This applies, contrary to a quite common belief [Diaz 2011], also to electric fields associated with a time-varying magnetic flux density (starting from the example of the wire and the loop, we could create, *mutatis mutandis*, a dual example for helical electric field lines).

A final issue to be discussed regards electromagnetic induction. It is well known that the description and interpretation of such a process may be not so trivial in some cases, especially in the presence of moving parts (the reader can develop an appreciation of this by consulting, among others, Hering [1908], Steinmetz [1908], Scanlon [1969], Munley [2004], Galili [2006], Redžić [2008], Rousseaux [2008], Giuliani [2010]). A revision of the problem is out of the scope of the present paper. However, if we consider the existence of non-closed $\boldsymbol{B}$-field lines winding around wires indefinitely, it is important to realize that a pictorial representation of such lines, relying on their physical existence, must be "handled with care" if used to evaluate the magnetic flux through a surface and, subsequently, to quantify electromagnetic induction. Concerning this, we remark that the integral form of Maxwell's equation for the curl of the electric field quantifies the phenomenon completely and unambiguously, provided that it is applied with respect to all the requirements given by Stokes' theorem and Leibniz's rule for differentiation under the integral sign. An excellent guide to this operation can be found in Auchmann [2014].

From a pedagogical perspective, in light of all previous discussions, it would be better to emphasize that the absence of magnetic monopoles is perfectly described by (1) and (2), without any need to infer general properties of the $\boldsymbol{B}$-field lines. These latter remain a useful illustrative tool (see their use in Belcher [2003], to support the intuitive analogy with the transmission of mechanical stress and pressure), provided that we remember the following rules:
-- $\boldsymbol{B}$-field lines are not always closed. Moreover, they may have a beginning or an end, in presence of singular points.
-- In the presence of lines that fill a region indefinitely, it is not possible to associate the line density with the field magnitude (unless one is able, in this region, to "count" each line only once). To preserve the correspondence, each line can be bounded between artificial cuts (creating starting and ending points). However, attention must be paid, because this trick may seem to violate equation (2).
-- The analysis of the flux through an open surface in terms of flux lines crossing the surface itself may be misleading. The mathematical definition of flux provides a direct and "safer" way of quantification.

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