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Abstract.
The conventional mass is a useful concept introduced to reduce the impact of the buoyancy correction in everyday mass measurements, thus avoiding in most cases its accurate determination, necessary in measurements of "true" mass. Although usage of conventional mass is universal and standardized, the concept is considered as a sort of second-choice tool, to be avoided in high-accuracy applications. In this paper we show that this is a false belief, by elucidating the role played by covariances between volume and mass and between volume and conventional mass at the various stages of the dissemination chain and in the relationship between the uncertainties of mass and conventional mass. We arrive at somewhat counter-intuitive results: the volume of the transfer standard plays a comparatively minor role in the uncertainty budget of the standard under calibration. In addition, conventional mass is preferable to mass in normal, in-air operation, as its uncertainty is smaller than that of mass, if covariance terms are properly taken into account, and the uncertainty over-stating (typically) resulting from neglecting them is less severe than that (always) occurring with mass. The same considerations hold for force. In this respect, we show that the associated uncertainty is the same using mass or conventional mass, and, again, that the latter is preferable if covariance terms are neglected.

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1. Introduction

In accurate mass measurements it is imperative that the air buoyancy effect be evaluated and corrected for.

In order to minimize the magnitude of the correction, in legal metrology the concept of *conventional mass value* is adopted [1] rather than that of mass. As a matter of fact, all the International Recommendations published by the International Organization of Legal Metrology (OIML) in the field of mass standards and weighing instruments refer to the conventional mass value. For example, the maximum permissible errors of mass standards [2, 3] and weighing instruments [4] as a function of their accuracy classes are given in terms of conventional mass. As a consequence, this concept is also useful in many technological and scientific applications, in that the correction is typically so small, compared to the target uncertainty, that it can be safely neglected. The conventional mass value is commonly perceived as a tool useful in low- to medium-accuracy applications, whereas in high-accuracy measurements mass is claimed to be preferable, in particular when the unknown standard has a density widely different from the reference density $\rho_c = 8\,000\,\mathrm{kg\,m^{-3}}$ and the measurement conditions deviate significantly from reference conditions. This view is reflected in the OIML Recommendation R111 [2], in which it is prescribed that, "If the air density deviates from $1.2\,\mathrm{kg\,m^{-3}}$ by more than 10 %, mass values should be used in calculations and the conventional mass should be calculated from the mass."

We will show that this suggestion does not hold, provided that covariances are taken into account. Indeed, contrary to what it is often assumed, the conventional mass value can fruitfully be employed also in high-accuracy applications, provided that the relevant buoyancy correction is applied and the associated uncertainty evaluated. Actually, in the same OIML document R111 a formula is also given for the buoyancy correction relevant to the conventional mass value. This correction, as it is well known, is much smaller than the corresponding correction for mass.

The correction, be it to mass or to conventional mass, plays a major role in the dissemination of mass standards. At each step of the traceability chain, the unknown standard is compared to a reference by means of a comparator. Following an earlier attempt to address the issue [5], it was shown for both mass [6] and conventional mass value [7] that along this chain the air buoyancy effect plays an opposite role in the correction of the measurement, depending on the role of the standard, i.e., depending on whether the standard is the unknown (during its calibration), or the reference (when used in the subsequent step). However, to the best of our knowledge the interplay between mass and conventional mass value along the dissemination chain has never been elucidated, with the consequence that a consistent propagation of uncertainties back and forth between these two closely connected quantities is lacking.

Often, in calibration certificates of mass standards, when both mass and conventional mass value are given, the same uncertainty (the one about mass) is associated to both estimates. This practice is in many cases an over-simplification of the correct procedure.

Similar considerations apply also to the calibration and subsequent use of weights for dead-weight force machines or pressure balances, or when calibrated weights are used as transfer standards to calibrate balances, as it has been recently demonstrated [8].

We show that the uncertainties associated with mass, conventional mass and force can be accurately evaluated only by taking into account the covariances related to air buoyancy, the most important being the one between mass (or conventional mass) and volume or density. This covariance constitutes a key term in the calibration of mass standards and in their subsequent use as reference standards of mass or force; and in understanding the relationship between the uncertainties associated with mass and conventional mass.

After a brief reminder about conventional mass in Section 2, in Section 3 we review the measurement models used in mass (and conventional mass) measurements and in the use of mass standards as force standards. We discuss and compare the associated uncertainties and covariance terms in Section 4. In Section 5 we give an example and in Section 6 we summarize our findings. An appendix is devoted to the expression of the most important formulas in terms of density.

2. Brief reminder

Two bodies, say, two mass standards i and j, having the same mass m_0, are in general not in equilibrium on a beam balance under the (locally constant) gravity acceleration g in air having density ρ_a, the net unbalance being $\rho_\mathrm{a}\,(V_i - V_j)$, where V denotes volume. Practicality suggests that standards having the same nominal value appear equal, i.e., balance each other, in air rather than in vacuum. This requirement is fulfilled when $m_i - m_j = \rho_\mathrm{a}\,(V_i - V_j)$. Indeed, mass standards having a given nominal value m_0 are not adjusted so as to have all a "true" mass $m_i = m_0$, in which case they would balance in vacuum, but, more practically, in such a way that they balance in air, specifically, in air having a reference density $\rho_0 = 1.2\,\mathrm{kg\,m^{-3}}$. As a consequence, the true mass m of a standard having nominal value m_0 is

$$m = m_0 + \rho_0\,(V - V_0)\,, \qquad (1)$$

where V_0 is a reference volume to be suitably specified in order to guarantee not only worldwide consistency among mass standards having nominal value m_0, but also traceability to the SI mass unit. Since volume is an extensive quantity, thus depending on the standard size and therefore on m_0, it is convenient to constrain the density m_0/V_0 to a specific value $m_0/V_0 = \rho_\mathrm{c} = 8\,000\,\mathrm{kg\,m^{-3}}$ at the reference temperature $t_\mathrm{ref} = 20\,^\circ\mathrm{C}$.

Therefore, a generic standard having nominal value m_0 is adjusted in such a way that its *conventional mass* (strictly speaking, the *conventional value of the result of weighing in air*) m_c, rather than its mass m, equals the nominal value, $m_\mathrm{c} = m_0$, from which equation (1) becomes

$$m = m_\mathrm{c}(1 - \rho_0/\rho_\mathrm{c}) + \rho_0 V, \qquad (2)$$

which in turn, using densities rather than volumes, becomes the familiar expression

$$m_c\,(1-\rho_0/\rho_c) = m\,(1-\rho_0/\rho)\,. \tag{3}$$

The conventional mass m_c of a mass standard is fully defined by equation (3) in terms of its mass m and density ρ, and the reference values ρ_c and ρ_0 for material and air, respectively [1]. The reference density $\rho_c = 8\,000\,\mathrm{kg\,m^{-3}}$, approximately corresponding to that of stainless steel, was adopted in 1971, the previous reference density being that of brass, $\rho = 8\,400\,\mathrm{kg\,m^{-3}}$ at the reference temperature of 0 °C [9]. The change reflects the corresponding change in the material generally used for the construction of mass standards. Up to 1971, mass standards had been adjusted so as to balance in air a brass reference, whereas since that date the reference was taken as a (ideal) stainless-steel standard. Incidentally, this implied a relative change in the adjustment of all standards and balances of $\approx 7\times10^{-6}$. This small revolution might be compared with the much smaller (relative) uncertainty component ($< 1\times10^{-7}$) initially associated with the intended redefinition of the kilogram. Yet, this uncertainty component raised deep concern [10], although it was recognized as irrelevant to the routine dissemination and calibration activity [11].

3. Mass, conventional mass and force

Routine calibrations of mass standards are aimed at determining their conventional mass. Accordingly, virtually all weighings give as output the *conventional value of the result of weighing in air*, i.e., the conventional mass of the body to be weighed. The conventional mass m_{ct} of the standard t under calibration is (we adopt the same notation as in [1])

$$m_{ct} = m_{cr} - \Delta m_{cw}, \tag{4}$$

where m_{cr} is the conventional mass of the reference standard r and $\Delta m_{cw} = m_{cwr} - m_{cwt}$ is the difference of the balance indications m_{cwr} and m_{cwt} when r and t are on its pan, respectively. In this respect, it is to be noted that the balance scale is adjusted in conventional mass.

Equation (4) is exact for a calibration performed at $\rho_0 = 1.2\,\mathrm{kg\,m^{-3}}$, with the reference standard at a temperature $t = t_{ref} = 20\,°\mathrm{C}$. In general, however, $\rho_a \neq \rho_0$, so that using equation (4) yields an approximate result. This approximation is inherent in the concept of conventional mass and is considered acceptable as far as the corresponding bias it not larger than 1/3 of the expanded uncertainty. This, combined with the fact that in the calibration of an OIML standard the expanded uncertainty has to be not larger than 1/3 of the corresponding maximum permissible error, implies that the bias has to be smaller than 1/9 of the maximum permissible error. A further general constraint on the density of OIML standards is that it must be such that a deviation of ±10 % of the air density from the reference value implies a corresponding bias not larger than 1/4 of the corresponding maximum permissible error. When one of the above-mentioned limits is exceeded, the correction has to be taken into account using a model more complete than (4).

To determine the correction C_{cb}, we use the well-known model for mass comparisons

$$m_{\mathrm{t}} = m_{\mathrm{r}} - C_{\mathrm{b}} - \Delta m_{\mathrm{w}}, \tag{5}$$

in which

$$C_{\mathrm{b}} = \rho_{\mathrm{a}}(V_{\mathrm{r}} - V_{\mathrm{t}}) \tag{6}$$

is the buoyancy correction for mass and Δm_{w} is the indication difference obtained from Δm_{cw} of equation (4) by using the balance sensitivity S which is, with good approximation [12], $S = (1 - \rho_0/\rho_{\mathrm{c}})^{-1}$, so that

$$\Delta m_{\mathrm{w}} = S^{-1}\Delta m_{\mathrm{cw}}. \tag{7}$$

Note that our model (5) departs from the corresponding model (9) in [1] as concerns the signs of the various contributions.

Using definition (2) in equation (5) yields

$$m_{\mathrm{ct}} = m_{\mathrm{cr}} - C_{\mathrm{cb}} - \Delta m_{\mathrm{cw}}, \tag{8}$$

where

$$C_{\mathrm{cb}} = (1 - \rho_0/\rho_{\mathrm{c}})^{-1}(\rho_a - \rho_0)(V_{\mathrm{r}} - V_{\mathrm{t}}) \tag{9}$$

is the counterpart of the corresponding term $\rho_{\mathrm{a}}(V_{\mathrm{r}} - V_{\mathrm{t}})$ in equation (5), and thus represents the sought buoyancy correction C_{cb} for conventional mass, and the last term in the RHS derives from equation (7). Equation (8) is the model for conventional mass comparisons, and thus constitutes the counterpart to equation (5).

Corrections (6) and (9) vanish for $\rho_{\mathrm{a}} = 0$ and $\rho_a - \rho_0 = 0$, respectively, or for $\Delta V = 0$.

Since in routine calibrations typically $\rho_{\mathrm{a}} - \rho_0 \approx 0$, the buoyancy correction C_{cb} to conventional mass is at least one order of magnitude smaller than the corresponding correction C_{b} for mass, so that the coefficient $(1 - \rho_0/\rho_{\mathrm{c}})^{-1} \approx 1.000\,15$ can be neglected in the evaluation of the correction uncertainty, as well as, in most cases, also in the evaluation of the correction itself.

Within this approximation, and using densities instead of volumes, the buoyancy correction C_{cb} for conventional mass takes the form

$$C_{\mathrm{cb}} = (\rho_a - \rho_0)\,(m_{\mathrm{r}}/\rho_{\mathrm{r}} - m_{\mathrm{t}}/\rho_{\mathrm{t}})\,, \tag{10}$$

or, assuming reasonably $m_{\mathrm{r}} \approx m_{\mathrm{t}} \approx m_{\mathrm{cr}}$,

$$C_{\mathrm{cb}} \approx m_{\mathrm{cr}}\,(\rho_a - \rho_0)\,(1/\rho_{\mathrm{r}} - 1/\rho_{\mathrm{t}})\,, \tag{11}$$

which corresponds to equation (10) in [1]. In this paper we take into account the coefficient in the calculation of the correction, but neglect it in the evaluation of the associated uncertainty.

Using definition (2) in different ways, equation (5) takes the equivalent forms

$$m_{\mathrm{ct}} = (1 - \rho_0/\rho_{\mathrm{c}})^{-1}[m_{\mathrm{r}} - \rho_{\mathrm{a}}(V_{\mathrm{r}} - V_{\mathrm{t}}) - \rho_0 V_{\mathrm{t}} - \Delta m_{\mathrm{w}}], \tag{12}$$

or

$$m_{\rm t} = (1 - \rho_0/\rho_{\rm c})m_{\rm cr} + \rho_0 V_{\rm r} - \rho_{\rm a}(V_{\rm r} - V_{\rm t}) - \Delta m_{\rm w}. \tag{13}$$

Equations (5), (8), (12) and (13) provide the mass $m_{\rm t}$ and the conventional mass $m_{\rm ct}$ of the standard t under calibration in terms of the mass $m_{\rm r}$ or conventional mass $m_{\rm cr}$ of the reference mass standard r.

As concerns the force $F_{\rm t}$ generated in the Earth gravitational field by the weight t having mass $m_{\rm t}$ and conventional mass $m_{\rm ct}$, it is

$$F_{\rm t} = (m_{\rm t} - \rho_{\rm aF} V_{\rm t})g = [m_{\rm ct}(1 - \rho_0/\rho_{\rm c}) + (\rho_0 - \rho_{\rm aF})V_{\rm t}]g, \tag{14}$$

where $\rho_{\rm aF}$ and g are the local air density and strength of the gravitational field, respectively, and the third member derives from definition (2) of conventional mass.

4. Uncertainties

4.1. uncertainties associated with mass and conventional mass

The uncertainty associated with $m_{\rm t}$ is obtained by applying the uncertainty propagation rule to equation (5):

$$u^2(m_{\rm t}) = u^2(m_{\rm r}) + u^2(C_{\rm b}) + u^2(\Delta m_{\rm w}), \tag{15}$$

and that associated with $m_{\rm ct}$ is obtained in the same way from equation (8):

$$u^2(m_{\rm ct}) = u^2(m_{\rm cr}) + u^2(C_{\rm cb}) + u^2(\Delta m_{\rm cw}), \tag{16}$$

In both equations, the second terms in the RHS, $u^2(C_{\rm b})$ and $u^2(C_{\rm cb})$, represent the uncertainty contributions due to the air buoyancy corrections for mass and conventional mass which, considering equations (6) and (9), are given by

$$u^2(C_{\rm b}) = \left(\frac{\partial m_{\rm t}}{\partial \rho_{\rm a}}\right)^2 u^2(\rho_{\rm a}) + \left(\frac{\partial m_{\rm t}}{\partial V_{\rm r}}\right)^2 u^2(V_{\rm r}) + \left(\frac{\partial m_{\rm t}}{\partial V_{\rm t}}\right)^2 u^2(V_{\rm t}) + 2\frac{\partial m_{\rm t}}{\partial m_{\rm r}}\frac{\partial m_{\rm t}}{\partial V_{\rm r}} u(m_{\rm r}, V_{\rm r}) + 2\frac{\partial m_{\rm t}}{\partial m_{\rm r}}\frac{\partial m_{\rm t}}{\partial \rho_{\rm a}} u(m_{\rm r}, \rho_{\rm a}) \tag{17}$$

and

$$u^2(C_{\rm cb}) = \left(\frac{\partial m_{\rm ct}}{\partial \rho_{\rm a}}\right)^2 u^2(\rho_{\rm a}) + \left(\frac{\partial m_{\rm ct}}{\partial V_{\rm r}}\right)^2 u^2(V_{\rm r}) + \left(\frac{\partial m_{\rm ct}}{\partial V_{\rm t}}\right)^2 u^2(V_{\rm t}) + 2\frac{\partial m_{\rm ct}}{\partial m_{\rm cr}}\frac{\partial m_{\rm ct}}{\partial V_{\rm r}} u(m_{\rm cr}, V_{\rm r}) + 2\frac{\partial m_{\rm ct}}{\partial m_{\rm cr}}\frac{\partial m_{\rm ct}}{\partial \rho_{\rm a}} u(m_{\rm cr}, \rho_{\rm a}), \tag{18}$$

respectively. Correlations between the volumes $V_{\rm t}$ and $V_{\rm r}$ have been neglected for simplicity, although they can occasionally occur.

The last terms in the RHS of equations (17) and (18) might be meaningful if the standard t is calibrated in the same laboratory in which the reference r was calibrated, i.e., when the correlation between the air densities $\rho_{\rm a}$ and $\rho_{\rm a1}$ measured during the two calibrations is non-negligible. These covariance terms are discussed in subsection 4.1.2.

The covariances $u(m_{\rm r}, V_{\rm r})$ and $u(m_{\rm cr}, V_{\rm r})$ arise from the calibration of the mass reference standard r. Previous papers [6, 7] have independently considered these two covariance terms. However, in order to clarify their role, these terms need to be jointly analysed and compared.

4.1.1. Covariances $u(m_{\rm r}, V_{\rm r})$ and $u(m_{\rm cr}, V_{\rm r})$ The covariances $u(m_{\rm r}, V_{\rm r})$ and $u(m_{\rm cr}, V_{\rm r})$ are determined on the basis of the previous calibration from which the value $m_{\rm r}$ was obtained using, say, a reference r1 having mass $m_{\rm r1}$ and volume $V_{\rm r1}$. The corresponding measurement models are

$$m_{\rm r} = m_{\rm r1} - \rho_{\rm a1}(V_{\rm r1} - V_{\rm r}) - \Delta m_{\rm w1} \tag{19}$$

and

$$m_{\rm cr} = (1 - \rho_0/\rho_{\rm c})^{-1}[m_{\rm r1} - \rho_{\rm a1}(V_{\rm r1} - V_{\rm r}) - \rho_0 V_{\rm r} - \Delta m_{\rm w1}], \tag{20}$$

where $\rho_{\rm a1}$ is the air density during the previous calibration.

From equations (19) and (20), it can be shown [6, 7] that

$$u(m_{\rm r}, V_{\rm r}) = \rho_{\rm a1} u^2(V_{\rm r}). \tag{21}$$

and

$$u(m_{\rm cr}, V_{\rm r}) = (\rho_{\rm a1} - \rho_0) u^2(V_{\rm r}), \tag{22}$$

respectively.

It follows that $u(m_{\rm r}, V_{\rm r}) - u(m_{\rm cr}, V_{\rm r}) = \rho_0 u^2(V_{\rm r})$. This implies that it is always $u(m_{\rm r}, V_{\rm r}) > u(m_{\rm cr}, V_{\rm r})$; in addition, knowing one of the two covariances is sufficient to obtain the other, with no need to know $\rho_{\rm a1}$.

Note that $u(m_{\rm r}, V_{\rm r}) \geq 0$, i.e, it is non-negative, and vanishes only if $m_{\rm r}$ has been determined in vacuum, (or if r is the international prototype of the kilogram itself, in the current implementation of the SI).

On the contrary, $u(m_{\rm cr}, V_{\rm r})$ is typically negative, taking positive values only in the rare cases in which $\rho_{\rm a1} > \rho_0$ and vanishing when $\rho_{\rm a1} = \rho_0$. This means that, as expected, $m_{\rm cr}$ is independent of $V_{\rm r}$ if the calibration of the reference standard was performed at $\rho_{\rm a1} = \rho_0$. Similarly, $m_{\rm r}$ is independent of $V_{\rm r}$ if the calibration was performed at $\rho_{\rm a1} = 0$.

Thus, the common practice of disregarding the covariance term $u(m_{\rm r}, V_{\rm r})$, implies the assumption that the weighing for the calibration of the reference standard was made in vacuum, whereas disregarding $u(m_{\rm cr}, V_{\rm r})$, implies the assumption $\rho_{\rm a1} = \rho_0$.

It is to be noted that $|u(m_{\rm r}, V_{\rm r})| = |u(m_{\rm cr}, V_{\rm r})|$ for $\rho_{\rm a1} = \rho_0/2$ whereas in general, in normal ambient conditions, where $\rho_{\rm a1} > \rho_0/2$, $|u(m_{\rm r}, V_{\rm r})| > |u(m_{\rm cr}, V_{\rm r})|$.

4.1.2. Covariances $u(m_{\rm r}, \rho_{\rm a})$ and $u(m_{\rm cr}, \rho_{\rm a})$ In those cases in which the air densities $\rho_{\rm a1}$ and $\rho_{\rm a}$ are correlated, equations (5) and (8) yield

$$u(m_{\rm r}, \rho_{\rm a}) = u(m_{\rm cr}, \rho_{\rm a}) = (V_{\rm r} - V_{\rm r1}) u(\rho_{\rm a}, \rho_{\rm a1}), \tag{23}$$

where we neglected the term $(1 - \rho_0/\rho_{\rm c})^{-1}$.

The contribution (23), usually negligible, may become significant when standards with very different volumes (or densities) are compared, as when silicon and stainless steel or platinum-iridium standards are involved in the same calibration.

4.1.3. Evaluation of $u^2(C_\mathrm{b})$ We are now able to evaluate the buoyancy contribution (17) to mass uncertainty. The covariance terms, considering equations (5), (6), (21) and (23), are

$$2\frac{\partial m_\mathrm{t}}{\partial m_\mathrm{r}}\frac{\partial m_\mathrm{t}}{\partial V_\mathrm{r}}u(m_\mathrm{r},V_\mathrm{r}) = 2(-\rho_\mathrm{a})u(m_\mathrm{r},V_\mathrm{r}) = -2\rho_\mathrm{a}\rho_\mathrm{a1}u^2(V_\mathrm{r}), \tag{24}$$

and

$$2\frac{\partial m_\mathrm{t}}{\partial m_\mathrm{r}}\frac{\partial m_\mathrm{t}}{\partial \rho_\mathrm{a}}u(m_\mathrm{r},\rho_\mathrm{a}) = 2(V_\mathrm{t}-V_\mathrm{r})(V_\mathrm{r}-V_\mathrm{r1})u(\rho_\mathrm{a},\rho_\mathrm{a1}), \tag{25}$$

so that

$$u^2(C_\mathrm{b}) = (V_\mathrm{r}-V_\mathrm{t})^2u^2(\rho_\mathrm{a}) + [(\rho_\mathrm{a}-\rho_\mathrm{a1})^2-\rho_\mathrm{a1}^2]u^2(V_\mathrm{r}) + \rho_\mathrm{a}^2u^2(V_\mathrm{t}) + 2(V_\mathrm{t}-V_\mathrm{r})(V_\mathrm{r}-V_\mathrm{r1})u(\rho_\mathrm{a},\rho_\mathrm{a1}), \tag{26}$$

the same result given in [6], apart from the last covariance term and a different factorization that makes the physical interpretation more straightforward. Equation (26) shows that the covariance term (24) has the effect of removing the contribution $\rho_\mathrm{a1}^2u^2(V_\mathrm{r})$ contained in $u^2(m_\mathrm{r})$ in equation (15), so that the net contribution of V_r to $u^2(m_\mathrm{t})$ is

$$u^2_{V_r}(m_\mathrm{t}) = (\rho_\mathrm{a}-\rho_\mathrm{a1})^2u^2(V_\mathrm{r}). \tag{27}$$

From equation (27) it is evident that the uncertainty associated with the volume of any transfer standard provides a negligible contribution (vanishing for $\rho_\mathrm{a}=\rho_\mathrm{a1}$) to the uncertainty of the mass of the standard under calibration, provided that the air densities ρ_a and ρ_a1 are not too different, as it is the case when the dissemination is carried out in the same laboratory. Indeed, an error in the volume of a mass standard involves of course a corresponding error in its mass when calibrating it. However, the error is compensated when that standard takes the role of reference in the subsequent calibration of other mass standards.

As concerns the covariance term (25) (the last term in equation (26)), it is worth noting that when working in the same laboratory typically $u(\rho_\mathrm{a1}) \approx u(\rho_\mathrm{a})$ and the two air densities are highly correlated. In such case, $u(\rho_\mathrm{a1},\rho_\mathrm{a}) \approx u^2(\rho_\mathrm{a})$. By developing calculations and considering the contribution $(V_\mathrm{r1}-V_\mathrm{r})^2u^2(\rho_\mathrm{a})$ already included into $u^2(m_\mathrm{r})$, it results that the net contribution of the air density to $u^2(m_\mathrm{t})$ is given by $(V_\mathrm{r1}-V_\mathrm{t})^2u^2(\rho_\mathrm{a})$, that is, it does not depend on the volume V_r. Hereafter, the covariance term (25) will be neglected.

When $u(m_\mathrm{r},V_\mathrm{r})=0$, i.e., when $\rho_\mathrm{a1}=0$, equation (26) becomes

$$u^2(C_\mathrm{b}) = \rho_\mathrm{a}^2u^2(V_\mathrm{r}) + \rho_\mathrm{a}^2u^2(V_\mathrm{t}) + (V_\mathrm{r}-V_\mathrm{t})^2u^2(\rho_\mathrm{a}). \tag{28}$$

This is also the expression resulting from neglecting covariance. This practice, inappropriate yet usually adopted, implies an overestimation of $u(m_\mathrm{t})$ (the sensitivity coefficient of the covariance being negative) due to double counting of the volume contribution, the overestimation amounting to $2\rho_\mathrm{a}\rho_\mathrm{a1}u^2(V_\mathrm{r})$.

4.1.4. Evaluation of $u^2(C_\mathrm{cb})$ To evaluate the buoyancy contribution (18) to the conventional mass uncertainty $u(m_\mathrm{ct})$, we use equation (22) in equation (18), thus obtaining

$$2\frac{\partial m_\mathrm{ct}}{\partial m_\mathrm{cr}}\frac{\partial m_\mathrm{ct}}{\partial V_\mathrm{r}}u(m_\mathrm{cr},V_\mathrm{r}) = -2(\rho_\mathrm{a}-\rho_0)u(m_\mathrm{cr},V_\mathrm{r}) = -2(\rho_\mathrm{a}-\rho_0)(\rho_\mathrm{a1}-\rho_0)u^2(V_\mathrm{r}) \quad (29)$$

for the covariance term, and eventually

$$u^2(C_\mathrm{cb}) = (V_\mathrm{r}-V_\mathrm{t})^2u^2(\rho_\mathrm{a}) + [(\rho_\mathrm{a}-\rho_\mathrm{a1})^2-(\rho_\mathrm{a1}-\rho_0)^2]u^2(V_\mathrm{r}) + (\rho_\mathrm{a}-\rho_0)^2u^2(V_\mathrm{t}). \quad (30)$$

We neglected the covariance (23), whose contribution is the same as for mass, and is given by equation (25).

Similarly to what was discussed in 4.1.3, this result is the same as that given in [7] (where densities are used rather than volumes) with a different factorization.

Equation (30) shows that the covariance term (29) has the effect of removing the contribution $(\rho_\mathrm{a1}-\rho_0)^2u^2(V_\mathrm{r})$ already contained in $u^2(m_\mathrm{cr})$, so that the net contribution of V_r to conventional-mass uncertainty $u^2(m_\mathrm{ct})$ is $(\rho_\mathrm{a}-\rho_\mathrm{a1})^2u^2(V_\mathrm{r})$, equal to the corresponding contribution to $u^2(m_\mathrm{t})$ given by equation (27).

When $\rho_\mathrm{a1}=0$, i.e., when $u(m_\mathrm{r},V_\mathrm{r})=0$, equation (30) becomes

$$u^2(C_\mathrm{cb}) = (V_\mathrm{r}-V_\mathrm{t})^2u^2(\rho_\mathrm{a}) + (\rho_\mathrm{a}^2-\rho_0^2)u^2(V_\mathrm{r}) + (\rho_\mathrm{a}-\rho_0)^2u^2(V_\mathrm{t}). \quad (31)$$

When $\rho_\mathrm{a1}=\rho_0$, i.e., when $u(m_\mathrm{cr},V_\mathrm{r})=0$, the covariance term (29) vanishes and equation (30) becomes

$$u^2(C_\mathrm{cb}) = (V_\mathrm{r}-V_\mathrm{t})^2u^2(\rho_\mathrm{a}) + (\rho_\mathrm{a}-\rho_0)^2u^2(V_\mathrm{r}) + (\rho_\mathrm{a}-\rho_0)^2u^2(V_\mathrm{t}). \quad (32)$$

Thus, the common-practice assumption $u(m_\mathrm{cr},V_\mathrm{r})=0$, leading to equation (32), has the consequence that the uncertainty $u(m_\mathrm{ct})$ is typically overestimated, since the covariance term (29) is typically negative (unless $\rho_\mathrm{a}<\rho_0<\rho_\mathrm{a1}$ or $\rho_\mathrm{a1}<\rho_0<\rho_\mathrm{a}$). However, the overestimation is less severe than in the case of mass, which constitutes an advantage of conventional mass over mass.

4.2. Comparison between $u(m_\mathrm{t})$ and $u(m_\mathrm{ct})$

The uncertainty $u(m_\mathrm{t})$ can be calculated from either model (5) or (13), using the buoyancy contribution (26) to obtain

$$u^2(m_\mathrm{t}) = u^2(m_\mathrm{r}) + [(\rho_\mathrm{a}-\rho_\mathrm{a1})^2-\rho_\mathrm{a1}^2]u^2(V_\mathrm{r}) + \rho_\mathrm{a}^2u^2(V_\mathrm{t}) + (V_\mathrm{r}-V_\mathrm{t})^2u^2(\rho_\mathrm{a}) + u^2(\Delta m_\mathrm{w}) \quad (33)$$

or, equivalently

$$u^2(m_t) = u^2(m_{cr}) + [(\rho_a - \rho_{a1})^2 - (\rho_{a1} - \rho_0)^2]u^2(V_r) + \rho_a^2 u^2(V_t) + (V_r - V_t)^2 u^2(\rho_a) + u^2(\Delta m_w). \quad (34)$$

In a similar manner, the uncertainties $u(m_{ct})$, obtained from models (8) and (12) using equation (30), are

$$u^2(m_{ct}) = u^2(m_{cr}) + (V_r - V_t)^2 u^2(\rho_a) + [(\rho_a - \rho_{a1})^2 - (\rho_{a1} - \rho_0)^2]u^2(V_r) + (\rho_0 - \rho_a)^2 u^2(V_t) + u^2(\Delta m_w) \quad (35)$$

or

$$u^2(m_{ct}) = u^2(m_r) + (V_r - V_t)^2 u^2(\rho_a) + [(\rho_a - \rho_{a1})^2 - \rho_{a1}^2]u^2(V_r) + (\rho_0 - \rho_a)^2 u^2(V_t) + u^2(\Delta m_w). \quad (36)$$

The relationship between the uncertainties associated with mass and conventional mass can be obtained directly from equation (2) in the form

$$u^2(m_{ct}) = u^2(m_t) + \rho_0^2 u^2(V_t) + 2\frac{\partial m_{ct}}{\partial m_t}\frac{\partial m_{ct}}{\partial V_t}u(m_t, V_t), \quad (37)$$

but is more easily obtained by subtraction of, say, equations (33) and (36):

$$u^2(m_{ct}) = u^2(m_t) + [(\rho_a - \rho_0)^2 - \rho_a^2]u^2(V_t). \quad (38)$$

Formula (37) is the correct tool to express the uncertainty associated with conventional mass in terms of the uncertainty about mass. Unfortunately, in common practice the covariance term (note that it is always negative) is ignored, which engenders errors and confusion.

Formula (38), which might be compared with formula (16) in [5], has a nice physical interpretation. In the uncertainty $u^2(m_{ct})$ about conventional mass, the contribution $(\rho_a - \rho_0)^2 u^2(V_t)$, specific of the buoyancy correction to conventional mass, is added to $u^2(m_t)$, whereas $\rho_a^2 u^2(V_t)$, specific of the buoyancy correction to mass, is subtracted.

The difference $\Delta u^2 = u^2(m_{ct}) - u^2(m_t)$ does not depend on $u(V_r)$, since $u(V_r)$ contributes in the same way (equation (27)) to the uncertainties (33) and (36) associated with mass and conventional mass. It rather depends on ρ_a, being equal to zero for $\rho_a = \rho_0/2 = 0.6\,\mathrm{kg\,m^{-3}}$. For $\rho_a = \rho_0$, $\Delta u^2 = -\rho_0^2 u^2(V_t)$ and for $\rho_a = 0$, $\Delta u^2 = \rho_0^2 u^2(V_t)$.

In the usual working conditions, i.e., for $\rho_a > 0.6\,\mathrm{kg\,m^{-3}}$, $u(m_{ct}) < u(m_t)$, that is, the uncertainty of the conventional mass is smaller than that of mass, provided that the covariance between mass and volume is properly taken into account.

Independent of the value of ρ_a, if $u(V_t)$ is small enough, as it is for example the case when the volumes are determined by hydrostatic weighing, $u(m_{ct}) \approx u(m_t)$.

4.3. Uncertainty associated with force

As concerns the force F_t generated by the artefact t according to equation (14), its uncertainty $u(F_t)$ must be the same, no matter whether it is derived from $u(m_t)$ or

$u(m_{\rm ct})$. To demonstrate this fact, covariance terms need to be taken into account, as for mass calibrations.

The uncertainty $u(F_{\rm t})$ is

$$u^2(F_{\rm t}) = \left[u^2(m_{\rm t}) + V_{\rm t}^2 u^2(\rho_{\rm aF}) + \rho_{\rm aF}^2 u^2(V_{\rm t}) + 2\frac{\partial F_{\rm t}}{\partial m_{\rm t}}\frac{\partial F_{\rm t}}{\partial V_{\rm t}} u(m_{\rm t}, V_{\rm t})\right] g^2 + \left(\frac{F_{\rm t}}{g}\right)^2 u^2(g), \quad (39)$$

or

$$u^2(F_{\rm t}) = \left[u^2(m_{\rm ct}) + V_{\rm t}^2 u^2(\rho_{\rm aF}) + (\rho_{\rm aF} - \rho_0)^2 u^2(V_{\rm t}) + 2\frac{\partial F_{\rm t}}{\partial m_{\rm ct}}\frac{\partial F_{\rm t}}{\partial V_{\rm t}} u(m_{\rm ct}, V_{\rm t})\right] g^2 + \left(\frac{F_{\rm t}}{g}\right)^2 u^2(g). \quad (40)$$

when expressed in terms of mass or conventional mass, respectively.

The covariance terms are

$$2\frac{\partial F_{\rm t}}{\partial m_{\rm t}}\frac{\partial F_{\rm t}}{\partial V_{\rm t}} u(m_{\rm t}, V_{\rm t}) = -2\rho_{\rm a}\rho_{\rm aF} u^2(V_{\rm t}), \quad (41)$$

from equation (21), and

$$\begin{aligned} 2\frac{\partial F_{\rm t}}{\partial m_{\rm ct}}\frac{\partial F_{\rm t}}{\partial V_{\rm t}} u(m_{\rm ct}, V_{\rm t}) &= -2(\rho_{\rm aF} - \rho_0) u(m_{\rm ct}, V_{\rm t}) \\ &= -2(\rho_{\rm aF} - \rho_0)(\rho_{\rm a} - \rho_0) u^2(V_{\rm t}), \end{aligned} \quad (42)$$

from equation (22).

Equations (39) and (40) thus become

$$u^2(F_{\rm t}) = \left\{u^2(m_{\rm t}) + V_{\rm t}^2 u^2(\rho_{\rm aF}) + [(\rho_{\rm aF} - \rho_{\rm a})^2 - \rho_{\rm a}^2] u^2(V_{\rm t})\right\} g^2 + \left(\frac{F_{\rm t}}{g}\right)^2 u^2(g). \quad (43)$$

and

$$u^2(F_{\rm t}) = \left\{u^2(m_{\rm ct}) + V_{\rm t}^2 u^2(\rho_{\rm aF}) + [(\rho_{\rm aF} - \rho_{\rm a})^2 - (\rho_{\rm a} - \rho_0)^2] u^2(V_{\rm t})\right\} g^2 + \left(\frac{F_{\rm t}}{g}\right)^2 u^2(g), \quad (44)$$

respectively.

By substitution of $u^2(m_{\rm ct})$ from equation (38) into equation (44) it can easily be verified that equations (43) and (44) are equivalent, as expected.

The effect of taking into account the covariance term is to reduce the uncertainty due to the air buoyancy. Apart from g, the contribution is the same as for mass, so that all the considerations made in section 4.1 apply here equally well.

This result is very important, because the volumes (or densities) of the weights used as force standards are usually estimated only approximately, with uncertainties typically much larger than those associated with volumes of mass standards. As a consequence, negliging the covariance term, as it is usually done (see, e.g., [13]), involves a large, unnecessary over-stating of the uncertainty associated with force. In addition, inconsistency is likely to arise between the uncertainty calculated from mass and that calculated from conventional mass. To the best of our knowledge, this aspect has never been analyzed in detail in force metrology, apart from brief discussions in [5, 14].

Using similar considerations, it has been demonstrated [8] that a balance calibrated in conventional mass can be used to determine the mass of an arbitrary sample with adequate uncertainty, provided that covariances are suitably taken into account.

5. Examples

In order to clarify the propagation of uncertainties through the dissemination chain in terms of mass and conventional mass, two examples are given. In both of them the tranfer standard r is a 1 kg standard having mass m_r with $u(m_\mathrm{r}) = 0.050\,\mathrm{mg}$ and $V_\mathrm{r} = 124.750(25)\,\mathrm{cm}^3$, which has been calibrated at $\rho_\mathrm{a1} = 1.190\,\mathrm{kg\,m^{-3}}$. The conventional mass uncertainty $u(m_\mathrm{cr})$ is, from equation (38), $u(m_\mathrm{cr}) = 0.040\,\mathrm{mg}$.

In the examples, this standard is used to calibrate at $\rho_\mathrm{a} = 1.150(1)\,\mathrm{kg\,m^{-3}}$ two working standards t1 (example 1) and t2 (example 2) having mass m_t1 and m_t2, respectively. The volume of t1, determined by hydrostatic weighing, is $V_\mathrm{t1} = 125.780(10)\,\mathrm{cm}^3$, whereas that of t2, estimated approximately considering a density $\rho = 7\,950(50)\,\mathrm{kg\,m^{-3}}$, is $V_\mathrm{t2} = 125.8(8)\,\mathrm{cm}^3$. The comparison uncertainty is $u(\Delta m_\mathrm{w}) = 0.010\,\mathrm{mg}$.

Subsequently, these two mass standards are used as transfer standards to calibrate at $\rho_\mathrm{a} = 1.120(1)\,\mathrm{kg\,m^{-3}}$ and with the same comparison uncertainty $u(\Delta m_\mathrm{w}) = 0.010\,\mathrm{mg}$, the same standard t3 having mass m_t3 and volume $V_\mathrm{t3} = 124.500(25)\,\mathrm{cm}^3$, determined by hydrostatic weighing. The two standards are also used as force standards at the air density $\rho_\mathrm{aF} = 1.120(1)\,\mathrm{kg\,m^{-3}}$, using $g = 9.81\,\mathrm{m\,s^{-2}}$ with negligible uncertainty for the acceleration due to gravity.

The uncertainties associated with mass, conventional mass and force are given in the last two columns and in the bottom line, respectively, of tables 1 and 2. The values in brackets are the uncertainties that would be obtained neglecting the covariance terms (24) and (29), as it is usually done. As concerns the force uncertainty, the two brackets refer to neglection of covariance in mass and conventional mass calculations, respectively.

In example 1 it is to be noted that, while $u(m_\mathrm{ct1}) > u(m_\mathrm{cr})$, the effect of covariance, jointly with the fact that $u(V_\mathrm{t1}) < u(V_\mathrm{r})$, is such that, surprisingly, $u(m_\mathrm{t1}) < u(m_\mathrm{r})$.

In example 2, the large uncertainty about the volume of t2 contributes considerably to the mass uncertainty $u(m_\mathrm{t2})$, whereas the conventional mass is much less affected.

As regards the calibration of t3, the covariance contribution is such that the uncertainty $u(m_\mathrm{t3})$ about its mass is largely independent of the uncertainty about the volume of the transfer standard used in the calibration, be it t1 (volume accurately known) or t2 (volume barely known).

A similar, less marked effect is observed on the uncertainty about conventional mass. Yet, the expanded uncertainty $U(m_\mathrm{ct3})$ obtained by properly considering covariance is $U(m_{ct3}) = 0.098\,\mathrm{mg}$. This value, despite the fact that the volume of the transfer standard t2 has a large uncertainty, is less than one fourth of the maximum permissible error (MPE) for 1 kg standards of OIML Class E1, thus largely complying with the requirement $U(m_c) < 0.167\,\mathrm{mg}$. For the same reason, i.e., the large uncertainty of V_{t2}, disregarding covariance would yield the non-complying value $U(m_{ct3}) = 0.18\,\mathrm{mg}$. This agrees with [7].

As concerns the forces generated by m_{t1} and m_{t2} it is worth noting that, although $u(m_{t1})$ and $u(m_{t2})$ are very different, the uncertainties related to the corresponding forces are comparable. In practice, taking into account the covariance contribution, the most important term for the uncertainty associated with the force is the contribution due to the air density uncertainty, whereas the volume does not contribute in a significant way. In addition, the over-stating of the force uncertainty that is given by neglecting covariance is much larger when working with mass than with conventional mass, which strongly indicates that, generally speaking, the latter route is preferable to the former.

Table 1. Dissemination from a reference (r) to a working (t3) standard through a transfer standard t1 whose volume is well-known. The last two columns give the uncertainties about mass and conventional mass, and the last row gives the uncertainties about the force generated by t1. The values in brackets are the (incorrect) uncertainties obtained neglecting the covariance terms. In the case of force, in the first bracket there is the uncertainty obtained when working with mass, in the second that obtained from conventional mass.

Standard	V/cm^3	$\rho_a/\mathrm{kg\,m^{-3}}$	$u(m,V)/\mathrm{mg\,cm^3}$	$u(m_c,V)/\mathrm{mg\,cm^3}$	$u(m)/\mathrm{mg}$	$u(m_c)/\mathrm{mg}$
m_r	124.750(25)	1.190	7.4×10^{-4}	-6.3×10^{-6}	0.050	0.040 (0.050)
m_{t1}	125.780(10)	1.150	1.2×10^{-4}	-5.0×10^{-6}	0.043 (0.060)	0.041 (0.051)
m_{t3}	125.500(25)	1.120	7.0×10^{-4}	-5.0×10^{-5}	0.051 (0.067)	0.043 (0.051)
@ $\rho_{aF} = 1.120(1)\,\mathrm{kg\,m^{-3}}$			$u(F_{t1}) = 13.0\,\mu\mathrm{N}$ (13.7 μN) (13.3 μN)			

Table 2. Dissemination from a reference (r) to a working (t3) standard through a transfer standard t2 whose volume is barely known. The last two columns give the uncertainties about mass and conventional mass, and the last row gives the uncertainties about the force generated by t2. The values in brackets are the (incorrect) uncertainties obtained neglecting the covariance terms. In the case of force, in the first bracket there is the uncertainty obtained when working with mass, in the second that obtained from conventional mass.

Standard	V/cm^3	$\rho_a/\mathrm{kg\,m^{-3}}$	$u(m,V)/\mathrm{mg\,cm^3}$	$u(m_c,V)/\mathrm{mg\,cm^3}$	$u(m)/\mathrm{mg}$	$u(m_c)/\mathrm{mg}$
m_r	124.750(25)	1.190	7.4×10^{-4}	-6.3×10^{-6}	0.050	0.040 (0.050)
m_{t2}	125.8(8)	1.150	7.4×10^{-1}	-3.2×10^{-2}	0.92 (0.92)	0.058 (0.065)
m_{t3}	125.500(25)	1.120	7.0×10^{-4}	-5.0×10^{-5}	0.056 (1.29)	0.049 (0.091)
@ $\rho_{aF} = 1.120(1)\,\mathrm{kg\,m^{-3}}$			$u(F_{t2}) = 13.2\,\mu\mathrm{N}$ (126.7 μN) (15.2 μN)			

6. Conclusions

The main conclusion of this paper is that, in order to maintain consistency along the whole dissemination chain, at each step the covariance terms arising from the buoyancy contribution to the uncertainties of both mass and conventional mass should be carefully taken into account. To enable the user to do so, it is recommended to give in calibration certificates the covariance $u(m_{\rm cr}, V_{\rm r})$ or $u(m_{\rm r}, V_{\rm r}))$ or, as in [6], $\rho_{\rm a1}$ and $u(V_{\rm r})$, or, of course, the corresponding values in terms of density (see the Appendix).

If calculations are carried out properly, somewhat surprising results are obtained. First, the uncertainty of the volume of the reference standard does not play a significant role. In addition, the uncertainty of a calibrated standard in some circumstances can be smaller that that of the reference. Finally, working with mass or conventional mass is essentially equivalent, although for normal operation in air the uncertainty of conventional mass is considerably smaller that that of mass. The relationship between mass and conventional mass uncertainties is governed by equation (38).

If covariance contributions are not taken into account, the uncertainty of mass suffers from a considerable over-stating, arising from a double counting of the uncertainty contribution of the volume of the reference standard. As to conventional mass, apart from a rarely occurring possibility of (moderate) uncertainty under-stating, in most practical situations the uncertainty over-stating is much smaller than for mass. This leads to the counter-intuitive conclusion that working with conventional mass is safer than working with mass.

The same conclusions are valid as concerns the uncertainty about the force generated by a deadweight. If calculations are carried out properly, working with mass or conventional mass is equivalent. This conclusion contradicts the widespread belief that high-level force metrology requires calibration in mass of the deadweight, thus limiting the conventional value to lower-level activity. Rather, conventional mass is preferable, being more tolerant towards neglection of covariance terms.

7. Appendix

The analysis given in this paper can be carried out in terms of density instead of volume. Here we give the most important formulas. Most of these results, some of which in a different form, are also given in [6, 7].

The weighing model (5) in terms of density is

$$m_{\rm t} = m_{\rm r}(1-\rho_{\rm a}/\rho_{\rm r})(1-\rho_{\rm a}/\rho_{\rm t})^{-1} - \Delta m_{\rm w} \tag{45}$$

or, using the conventional mass $m_{\rm cr}$,

$$m_{\rm t} = m_{\rm cr}\frac{(1-\rho_0/\rho_{\rm c})(1-\rho_{\rm a}/\rho_{\rm r})}{(1-\rho_0/\rho_{\rm r})(1-\rho_{\rm a}/\rho_{\rm t})} - \Delta m_{\rm w}, \tag{46}$$

which is equivalent to equation (13).

For the conventional mass (8), under the usual assumption $m_{\rm t} \approx m_{\rm r}$,

$$m_{\rm ct} = m_{\rm cr}\left[1 + (\rho_{\rm a} - \rho_0)\left(1/\rho_{\rm t} - 1/\rho_{\rm r}\right)\right] - \Delta m_{\rm cw}, \tag{47}$$

or

$$m_{\rm ct} = m_{\rm r}\frac{(1-\rho_0/\rho_{\rm r})}{(1-\rho_0/\rho_{\rm c})}\left[1 + (\rho_{\rm a} - \rho_0)\left(1/\rho_{\rm t} - 1/\rho_{\rm r}\right)\right] - \Delta m_{\rm cw}. \tag{48}$$

The counterparts to the covariances (21) and (22) are

$$u(m_{\rm r}, \rho_{\rm r}) = -\rho_{\rm a1}\frac{m_{\rm r}}{\rho_{\rm r}^2}u^2(\rho_{\rm r}) \tag{49}$$

and, taking $m_{\rm cr1} \approx m_{\rm cr}$,

$$u(m_{\rm cr}, \rho_{\rm r}) = -(\rho_{\rm a1} - \rho_0)\frac{m_{\rm cr}}{\rho_{\rm r}^2}u^2(\rho_{\rm r}), \tag{50}$$

respectively.

By pairwise comparing equations (49) *vs* (21) and (50) *vs* (22), and considering that typically $u(V)/V \approx u(\rho)/\rho$, the useful formulas

$$u(m, \rho) = -\frac{m}{V^2}u(m, V) \tag{51}$$

and

$$u(m_{\rm c}, \rho) = -\frac{m}{V^2}u(m_{\rm c}, V) \tag{52}$$

are obtained.

The buoyancy contribution to the mass uncertainty, which in terms of volume is given by equation (26), becomes

$$u^2(C_{\rm b}) = \left[m_{\rm r}\left(\frac{1}{\rho_{\rm t}} - \frac{1}{\rho_{\rm r}}\right)\right]^2 u^2(\rho_{\rm a}^2) + \rho_{\rm a}^2\frac{m_{\rm r}^2}{\rho_{\rm t}^4}u^2(\rho_{\rm t}) + \left[(\rho_{\rm a} - \rho_{\rm a1})^2 - \rho_{\rm a1}^2\right]\frac{m_{\rm r}^2}{\rho_{\rm r}^4}u^2(\rho_{\rm r}), \tag{53}$$

whereas the corresponding contribution to conventional mass, equation (30), is

$$u^2(C_{\rm cb}) = \left[m_{\rm cr}\left(\frac{1}{\rho_{\rm t}} - \frac{1}{\rho_{\rm r}}\right)\right]^2 u^2(\rho_{\rm a}^2) + (\rho_{\rm a} - \rho_0)^2\frac{m_{\rm cr}^2}{\rho_{\rm t}^4}u^2(\rho_{\rm t}) + \left[(\rho_{\rm a} - \rho_{\rm a1})^2 - (\rho_{\rm a1} - \rho_0)^2\right]\frac{m_{\rm cr}^2}{\rho_{\rm r}^4}u^2(\rho_{\rm r}). \tag{54}$$

Equation (38) can be expressed in terms of density as

$$u^2(m_{\rm ct}) = u^2(m_{\rm t}) - \left[(\rho_{\rm a} - \rho_0)^2 - \rho_{\rm a}^2\right]\frac{m_{\rm t}^2}{\rho_{\rm t}^4}u^2(\rho_{\rm t}). \tag{55}$$

As concerns force, equations (14) become here

$$F_{\rm t} = m_{\rm t}(1 - \rho_{\rm aF}/\rho_{\rm t})g \tag{56}$$

and

$$F_{\rm t} = m_{\rm ct}(1 - \rho_0/\rho_{\rm c})(1 - \rho_0/\rho_{\rm t})^{-1}(1 - \rho_{\rm aF}/\rho_{\rm t})g \approx m_{\rm ct}\left[1 + (\rho_0 - \rho_{\rm aF})/\rho_{\rm t} - (\rho_0/\rho_{\rm c})\right]g, \tag{57}$$

The last term, a first-order approximation, is the same as equation (3) in [13].

The corresponding (equivalent) uncertainties are

$$u^2(F_\mathrm{t}) = \left[u^2(m_\mathrm{t}) + \frac{m_\mathrm{t}^2}{\rho_\mathrm{t}^2}u^2(\rho_\mathrm{aF}) + \frac{m_\mathrm{t}^2}{\rho_\mathrm{t}^4}[(\rho_\mathrm{aF} - \rho_\mathrm{a})^2 - \rho_\mathrm{a}^2]u^2(\rho_\mathrm{t})\right] g^2 + \left(\frac{F_\mathrm{t}}{g}\right)^2 u^2(g) \tag{58}$$

and

$$u^2(F_\mathrm{t}) = \left[u^2(m_\mathrm{ct}) + \frac{m_\mathrm{ct}^2}{\rho_\mathrm{t}^2}u^2(\rho_\mathrm{aF}) + \frac{m_\mathrm{ct}^2}{\rho_\mathrm{t}^4}[(\rho_\mathrm{aF} - \rho_\mathrm{a})^2 - (\rho_\mathrm{a} - \rho_0)^2]u^2(\rho_\mathrm{t})\right] g^2 + \left(\frac{F_\mathrm{t}}{g}\right)^2 u^2(g), \tag{59}$$

respectively.

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