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# Uncertainty in discrete-time integration — The case of static gas meters

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ABSTRACT

We consider the evaluation of uncertainty in a particular case of discrete-time integration, i.e., that of static gas meters. We show that the current approach, which does not take correlations into account, can lead to underrating of the uncertainty associated with the estimate of the delivered volume. Now, a correct evaluation of uncertainty in the case e.g. of online measurements along large offshore pipelines is important in order to remove technical barriers to international trade.

We focus our discussion on the practical example of gas meters, but the framework of uncertainty evaluation we provide is valid in general for all those measurements involving discrete-time integration, and has therefore a wide applicability.

## 1. Introduction

Discrete-time integration is used to obtain the value of the integral of a quantity from a set of measured values of the quantity. Various methods are available, such as the rectangular approximation (order 0), the trapezoidal rule (order 1) or the Runge–Kutta method (orders 2 and higher, depending on the parameters) [1,2].

These methods are widely used in numerical analysis for a range of applications, typically aimed at the computation of the time evolution of a quantity, rather than at the computation of the integral of the area under a curve [3].

Recent developments in the field of flow metering, though, revived interest in the computation of the integral area under a curve which is not known analytically but only at discrete points, acquired at successive time instants. An important application concerns the development of digital, or static gas meters [4,5] as an alternative to positive-displacement gas meters, which are based on the mechanical, cyclic displacement of a suitable element under the effect of the gas flow, and on the counting of the resulting cyclic volumes.

Indeed, positive-displacement instruments are still dominating the field, especially as concerns fiscal gas metering. Recent evolutions of the relevant normative [6,7], though, prompted the development of the novel class of gas meters, which are based on some kind of continuous gas flowrate sensor. Flowrate values are acquired at time instants whose separation $\delta t$ is nominally constant, and the accumulated volume is thus obtained by discrete-time integration of these values, usually applying the trapezoidal rule (see Eq. (2)).

This technology constitutes a true change of paradigm. On the one hand, it overcomes the old problem of low flowrates for positive-displacement meters, both in their calibration and in measurement. In the former case, a large time is required in order to accumulate a sufficient number of cyclic volumes. In the latter, a short measurement time implies a large measurement uncertainty. Another well-known problem is the change in the value of cyclic volume with ageing of the instrument. On the other hand, the calibration of static meters requires a minimum test time which, at higher flow rates, introduces new problems due to the corresponding large accumulated volume [8].

This change of paradigm implies modifications in calibration and usage procedures. The technology is still at a developing stage and it can be anticipated that will reach full maturity in a few years. The relevant normative is being updated as regards the fiscal and operational aspects of the deployment of static gas meters [6,7]. These updates are based mainly on field experience gained by the first pioneering installations and studies [5,9] and thus, although correct, lack a coherent metrological framework. For instance, the assessment of the metrological performance of a gas meter is currently limited to the prescription of a maximum permissible error [6]; however, in modern approaches to conformity assessment the role of uncertainty is of paramount importance [10], but the normative provides no indications about the way of determining uncertainty. In particular no mention is made, in the bibliography we have analyzed (see, as an example, [11,12]), of the uncertainty introduced by the operation of integration.

We therefore consider it useful to develop a rigorous evaluation of the uncertainty introduced by the mathematical process of integration as a component of the uncertainty associated with the volume estimates provided by the digital gas meters which are being developed and deployed [13]. Laboratories for testing and calibration of static gas meters

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usually apply uncertainty analysis for uncorrelated input quantities. As we will show in the following, this approach is inadequate.

We focus our uncertainty analysis on discrete-time integration using the trapezoidal rule. Our choice is dictated by the fact that this rule is, to the best of our knowledge, almost universally adopted in current static gas meters, the reason being that it represents a good compromise between accuracy and economy of requirements in terms of memory and computational power, thus allowing a compact and cheap realization of the instrument.

Technological and normative developments strongly suggest that static metering will soon be extended to water meters and to several other applications connected to digitalization of the metrological infrastructure. More in general, digital acquisition and the development of the so-called "Internet of Things" (IoT) (see, *e.g.*, [14,15]) will call for a generalized application of numerical integration schemes for data acquisition from instruments distributed across the territory. The aforementioned uncertainty evaluation will therefore be required also for many other applications. Accordingly, and although in the paper we focus on gas volume measurement, our discussion is valid in general for all those measurements involving discrete-time integration by means of the trapezoidal rule.[1] The framework of uncertainty evaluation we provide here has therefore a wide applicability.

The paper is organized as follows: in Section 2 we establish the measurement model, in Section 3 we evaluate the uncertainty associated with the (estimated) delivered volume, we give formulæfor both uncorrelated 3.1 and correlated 3.3 input quantities, and compare the former with the current practice 3.2. In Section 4 we give an example based on the calibration of a domestic gas meter and show the dramatic underrating of the uncertainty provided by the current practice. In Section 5 we provide some comments and in Section 6 we draw some conclusions.

## 2. Measurement model

As it is well known, in fluid mechanics the flowrate $Q_V$ ($Q_m$) is defined as the time derivative of the volume (mass) of fluid flowing through a given section. Knowing the flowrate as a function of time, the volume $V_t$ and the mass $m_t$ of the fluid accumulated at time $t$ after an initial time $t_0$ are given by:

$$V_t = \int_{t_0}^{t} Q_V(t)\mathrm{d}t \quad \text{and} \quad m_t = \int_{t_0}^{t} Q_m(t)\mathrm{d}t \tag{1}$$

respectively.

The quantity of interest being here volume, we consider the first of Eqs. (1) (the treatment of the second would be equivalent) and drop hereafter the subscript $V$ from $Q_V$.

As discussed in Section 1, in actual applications $N+1$ flowrate values $Q_j$, the *indications*, are acquired at discrete time instants $t_j$ over a time interval $t - t_0 = \Delta t$. According to the trapezoidal rule, the integral (1) is then discretized as follows:

$$V_t = \sum_{i=1}^{N} V_i \approx \sum_{i=1}^{N} \overline{Q}_i \delta t_i, \tag{2}$$

where $\overline{Q}_i = \frac{Q_j + Q_{j-1}}{2}$ and $\delta t_i = t_j - t_{j-1}$ ($\sum \delta t_i = \Delta t$) are the $N$ individual average flowrates and the corresponding time intervals, respectively. Hereafter, we replace the $\approx$ symbol with the equality for simplicity.

As it can be seen, the approximation is of order 1, indicating that the (unknown) function $Q(t)$ is approximated by straight segments, each of which shares its endpoints with the previous and the next segments.

Eq. (2) represents the measurement model. In it, the $Q_j$ and $t_j$ are the $2(N+1)$ input quantities and $V_t$ is the output quantity, the measurand.

[1] The analysis can easily be extended to other integration schemes by modifying the measurement model (2) and, consequently, the matrices $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ (3).

## 3. Uncertainty evaluation

Model (2) can be viewed as an instance of a multi-stage measurement model. The measurand $V_t$ is a function of the volumes $V_i$ which, in turn, are functions of the $N$ individual average flowrates $\overline{Q}_i$ and the $N$ corresponding time intervals $\delta t_i$ which, in turn, are functions of the $N+1$ flowrates $Q_j$ and the $N+1$ time instants $t_j$. The volumes $V_i$ cannot be considered independent as each $Q_j$ and $t_j$ both contribute to the adjacent $V_{i-1}$ and $V_i$ volumes.

Matrix formalism is needed to appropriately capture the multivariate nature of the measurement model.

The volume $V_t$ is thus written as $V_t = f(\boldsymbol{A})$, where $\boldsymbol{A}^\top = \left[\boldsymbol{Q}^\top \; \boldsymbol{t}^\top\right]$ is a $2(N+1)$ block vector with $\boldsymbol{Q} = (Q_0, \ldots, Q_N)^\top$ and $\boldsymbol{t} = (t_0, \ldots, t_N)^\top$.

To express the function $f$, we define:

$$\boldsymbol{X}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \ddots & 1 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix}_{N\times(N+1)}$$
$$\boldsymbol{X}_2 = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \vdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \ddots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{N\times(N+1)}, \tag{3}$$

so that the measurement model (2) can be expressed in matrix form as:

$$V_t = \left[\boldsymbol{X}_1\boldsymbol{Q}\right]^\top \boldsymbol{X}_2\boldsymbol{t} = \overline{\boldsymbol{Q}}^\top \boldsymbol{\delta t}, \tag{4}$$

where we have introduced the vectors $\overline{\boldsymbol{Q}} = \boldsymbol{X}_1\boldsymbol{Q} = \left(\overline{Q}_1, \ldots, \overline{Q}_N\right)^\top$ and $\boldsymbol{\delta t} = \boldsymbol{X}_2\boldsymbol{t} = \left(\delta t_1, \ldots, \delta t_N\right)^\top$ of the average flowrates and the corresponding time intervals, respectively.

The (squared) standard uncertainty of $V_t$, $u^2\left(V_t\right)$, is obtained from the covariance matrix $\boldsymbol{U}_{\boldsymbol{A}}$ of the input vector $\boldsymbol{A}$ using the well-known standard procedure described in [16]:

$$u^2\left(V_t\right) = \boldsymbol{J}_{\boldsymbol{A}}\boldsymbol{U}_{\boldsymbol{A}}\boldsymbol{J}_{\boldsymbol{A}}^\top, \tag{5}$$

where $\boldsymbol{J}_{\boldsymbol{A}} = \left[\partial V_t/\partial A_0, \ldots, \partial V_t/\partial A_{(2N+2)}\right]$ is the Jacobian, or sensitivity matrix, in this case a row vector. It is obviously $\boldsymbol{J}_{\boldsymbol{A}} = \left[\boldsymbol{J}_Q \; \boldsymbol{J}_t\right]$.

As concerns the covariance matrix of $\boldsymbol{A}$, we make the reasonable assumption that the observations of flowrate and time instants are independent, which means that

$$\boldsymbol{U}_{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{U}_Q & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_t \end{bmatrix}, \tag{6}$$

where $\boldsymbol{U}_Q$ and $\boldsymbol{U}_t$ are the covariance matrices of $\boldsymbol{Q}$ and $\boldsymbol{t}$, respectively. Expression (5) thus becomes

$$u^2(V_t) = \left[\boldsymbol{J}_Q \; \boldsymbol{J}_t\right]\begin{bmatrix} \boldsymbol{U}_Q & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_t \end{bmatrix}\begin{bmatrix} \boldsymbol{J}_Q^\top \\ \boldsymbol{J}_t^\top \end{bmatrix} = \boldsymbol{J}_Q\boldsymbol{U}_Q\boldsymbol{J}_Q^\top + \boldsymbol{J}_t\boldsymbol{U}_t\boldsymbol{J}_t^\top. \tag{7}$$

It can easily be verified that $\boldsymbol{J}_Q = \boldsymbol{t}^\top\boldsymbol{X}_2^\top\boldsymbol{X}_1$ and $\boldsymbol{J}_t = \boldsymbol{Q}^\top\boldsymbol{X}_1^\top\boldsymbol{X}_2$.

Introducing in expression (7) the covariance matrices $\boldsymbol{U}_{\overline{Q}} = \boldsymbol{X}_1\boldsymbol{U}_Q\boldsymbol{X}_1^\top$ and $\boldsymbol{U}_{\delta t} = \boldsymbol{X}_2\boldsymbol{U}_t\boldsymbol{X}_2^\top$ we obtain

$$u^2(V_t) = \overline{\boldsymbol{Q}}^\top\boldsymbol{U}_{\delta t}\overline{\boldsymbol{Q}} + \boldsymbol{\delta t}^\top\boldsymbol{U}_{\overline{Q}}\boldsymbol{\delta t}, \tag{8}$$

an equivalent and perhaps more elegant way to write expression (7).

Eq. (8) gives the (squared) standard uncertainty associated with the volume indicated by the gas meter after the time $\Delta t$, given the measurement model (4) and the $2(N+1)$-component input vector $\boldsymbol{A}^\top = \left[\boldsymbol{Q}^\top \; \boldsymbol{t}^\top\right]$.

### 3.1. Uncorrelated input quantities

It is interesting to calculate the covariance matrices $\boldsymbol{U}_{\overline{Q}}$ and $\boldsymbol{U}_{\delta t}$ when the $Q_i$, as well as the $t_i$, are uncorrelated. To demonstrate our argument, we consider the realistic case in which the indicated flowrates, as well as the time instants, have the same standard uncertainties, $u_Q$ and $u_t$, respectively. Therefore, $\boldsymbol{U}_Q = u_Q^2 \boldsymbol{I}$, $\boldsymbol{U}_t = u_t^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is the unitary matrix. With these assumptions,

$$\boldsymbol{U}_{\overline{Q}} = u_Q^2 \boldsymbol{X}_1 \boldsymbol{X}_1^\top \quad \text{and} \quad \boldsymbol{U}_{\delta t} = u_t^2 \boldsymbol{X}_2 \boldsymbol{X}_2^\top \tag{9}$$

It comes out that

$$\boldsymbol{U}_{\overline{Q}} = \frac{u_Q^2}{2} \begin{bmatrix} 1 & 1/2 & 0 & \cdots & 0 \\ 1/2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 1/2 \\ 0 & \cdots & 0 & 1/2 & 1 \end{bmatrix} \tag{10}$$

and

$$\boldsymbol{U}_{\delta t} = 2u_t^2 \begin{bmatrix} 1 & -1/2 & 0 & \cdots & 0 \\ -1/2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & -1/2 \\ 0 & \cdots & 0 & -1/2 & 1 \end{bmatrix}. \tag{11}$$

Expressions (10) and (11) show that any two adjacent average flowrates are positively correlated, and any two adjacent time intervals are negatively correlated even in the case of uncorrelated input quantities, a finding in agreement with common sense. The correlation coefficients are $\rho(\overline{Q}_i, \overline{Q}_{i+1}) = 1/2$ and $\rho(\delta t_i, \delta t_{i+1}) = -1/2$, respectively, $\forall i < N$.

Eq. (8) thus becomes

$$u^2(V_t) = \left[ \sum_{i=1}^{N} \overline{Q}_i^2 - \sum_{i=1}^{N-1} \overline{Q}_i \overline{Q}_{i+1} \right] u_{\delta t}^2 + \left[ \sum_{i=1}^{N} \delta t_i^2 + \sum_{i=1}^{N-1} \delta t_i \delta t_{i+1} \right] u_{\overline{Q}}^2. \tag{12}$$

Although the specific form of Eqs. (10) to (12) holds strictly only for the case of equal standard uncertainties, it is true in general that, as expected, the negative covariances in $\boldsymbol{U}_{\delta t}$ tend to reduce the uncertainty component due to $t$, whereas those in $\boldsymbol{U}_{\overline{Q}}$, being positive, contribute significantly to the component due to $Q$.

Neglecting the covariance terms in Eq. (12), one obtains the same formula that would be provided by the law of propagation of uncertainties for uncorrelated input quantities, as given by expression (10) in [17], applied to model (2):

$$u^{2*}(V_t) = \sum_{i=1}^{N} \overline{Q}_i^2 u_{\delta t}^2 + \sum_{i=1}^{N} \delta t_i^2 u_{\overline{Q}}^2. \tag{13}$$

This is the expression currently used for the uncertainty evaluation of the volume indicated by a static gas meter and, possibly, in many other measurements involving discrete-time integration.

### 3.2. Comparison with the current practice

To gain some insight into the quantitative difference between the uncertainties obtained by keeping into account or neglecting covariances – Eqs. (12) and (13), respectively – we consider that the time intervals $\delta t$ are nominally equal and further assume that the flowrate $Q$ is approximately constant during the time interval $\Delta t$.

In this case, Eqs. (12) and (13) become

$$u^2(V_t) \approx \overline{Q}^2 u_{\delta t}^2 + (2N-1)\delta t^2 u_{\overline{Q}}^2 \tag{14}$$

and

$$u^{2*}(V_t) \approx N \left[ \overline{Q}^2 u_{\delta t}^2 + \delta t^2 u_{\overline{Q}}^2 \right], \tag{15}$$

respectively.

It is worth obtaining the corresponding expression for the relative (squared) standard uncertainties $u_{\mathrm{rel}}^2(V_t)$ and $u_{\mathrm{rel}}^{2*}(V_t)$. Recalling that $V_t \approx N\overline{Q}\delta t$ one has

$$u_{\mathrm{rel}}^2(V_t) \approx \frac{1}{N^2} \left[ u_{\mathrm{rel}}^2(\delta t) + (2N-1) u_{\mathrm{rel}}^2(\overline{Q}) \right] \tag{16}$$

and

$$u_{\mathrm{rel}}^{2*}(V_t) \approx \frac{1}{N} \left[ u_{\mathrm{rel}}^2(\delta t) + u_{\mathrm{rel}}^2(\overline{Q}) \right]. \tag{17}$$

Expression (17) is also given in Appendix A.1. of Ref. [8].

The ratio between Eqs. (16) and (17) is

$$R^2 = u_{\mathrm{rel}}^2(V_t)/u_{\mathrm{rel}}^{2*}(V_t) = \frac{1}{N} \left[ 1 + \frac{2(N-1)}{1 + u_{\mathrm{rel}}^2(\delta t)/u_{\mathrm{rel}}^2(\overline{Q})} \right]. \tag{18}$$

The (square root of the) monotonic function (18) reaches its minimum $R_{\min} = 1/\sqrt{N}$ for $u_{\mathrm{rel}}^2(\delta t) \gg u_{\mathrm{rel}}^2(\overline{Q})$, and its maximum $R_{\max} = \sqrt{2 - 1/N}$ for $u_{\mathrm{rel}}^2(\delta t) \ll u_{\mathrm{rel}}^2(\overline{Q})$. Indeed, we can hardly conceive situations in which $u_{\mathrm{rel}}^2(\delta t)/u_{\mathrm{rel}}^2(\overline{Q}) > 4$, so that in practice $0.5 < R < 1.41$. Therefore, neglecting covariances, as done in the current uncertainty evaluation, can either underrate or overrate the uncertainty by significant amounts. Whilst the significance of such mistaken uncertainties depends on the application, yet a correct uncertainty evaluation should always be preferred to a flawed one.

Fig. 1 gives a diagram of $R$ as a function of $u_{\mathrm{rel}}^2(\delta t)/u_{\mathrm{rel}}^2(\overline{Q})$ and for different values of $N$. The function is practically independent of $N$ for $N > 20$, that is, in most practical cases.

### 3.3. Correlated input quantities

In a different *scenario*, for example when the gas meter, after calibration, is used online, the $Q_i$ are correlated by an uncertainty component due to calibration. To simplify the treatment, we assume that the uncertainty is approximately the same for all the indications $Q_i$ and has the form $u_Q^2 = u_{\mathrm{r}}^2 + u_{\mathrm{c}}^2$, the subscripts r and c standing for 'random' or 'repeatability' and 'calibration', respectively.[2]

In this case the covariance matrix $\boldsymbol{U}_Q$ is written as

$$\boldsymbol{U}_Q = \begin{bmatrix} u_{\mathrm{r}}^2 + u_{\mathrm{c}}^2 & u_{\mathrm{c}}^2 & \cdots & u_{\mathrm{c}}^2 \\ u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2 + u_{\mathrm{c}}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_{\mathrm{c}}^2 \\ u_{\mathrm{c}}^2 & \cdots & u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2 + u_{\mathrm{c}}^2 \end{bmatrix}, \tag{19}$$

or

$$\boldsymbol{U}_Q = u_{\mathrm{r}}^2 \boldsymbol{I} + u_{\mathrm{c}}^2 \mathbf{1}\mathbf{1}^\top, \tag{20}$$

where $\boldsymbol{I}$ is the identity matrix and $\mathbf{1}^\top = [1, 1, \ldots, 1]_{N+1}$.

Similar considerations hold for $\boldsymbol{U}_t$, although the calibration uncertainty of the timer is usually negligible in real applications and will therefore not be considered here for simplicity.

With the covariance matrix $\boldsymbol{U}_Q$ defined by expression (20), the corresponding $\boldsymbol{U}_{\overline{Q}}$ is

$$\boldsymbol{U}_{\overline{Q}} = u_{\mathrm{r}}^2 \boldsymbol{X}_1 \boldsymbol{X}_1^\top + u_{\mathrm{c}}^2 \boldsymbol{X}_1 \mathbf{1} \left[ \boldsymbol{X}_1 \mathbf{1} \right]^\top = u_{\mathrm{r}}^2 \boldsymbol{X}_1 \boldsymbol{X}_1^\top + 4u_{\mathrm{c}}^2 \mathbf{1}\mathbf{1}^\top, \tag{21}$$

or

$$\boldsymbol{U}_{\overline{Q}} = \begin{bmatrix} u_{\mathrm{r}}^2/2 + 4u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2/4 + 4u_{\mathrm{c}}^2 & 4u_{\mathrm{c}}^2 & \cdots & 4u_{\mathrm{c}}^2 \\ u_{\mathrm{r}}^2/4 + 4u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2/2 + 4u_{\mathrm{c}}^2 & \ddots & \ddots & \vdots \\ 4u_{\mathrm{c}}^2 & \ddots & \ddots & \ddots & 4u_{\mathrm{c}}^2 \\ \vdots & \ddots & \ddots & u_{\mathrm{r}}^2/2 + 4u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2/4 + 4u_{\mathrm{c}}^2 \\ 4u_{\mathrm{c}}^2 & \cdots & 4u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2/4 + 4u_{\mathrm{c}}^2 & u_{\mathrm{r}}^2/2 + 4u_{\mathrm{c}}^2 \end{bmatrix}. \tag{22}$$

[2] We neglect here for simplicity of notation the uncertainty component due to finite resolution, which by the way is typically negligible (see also 4.1).

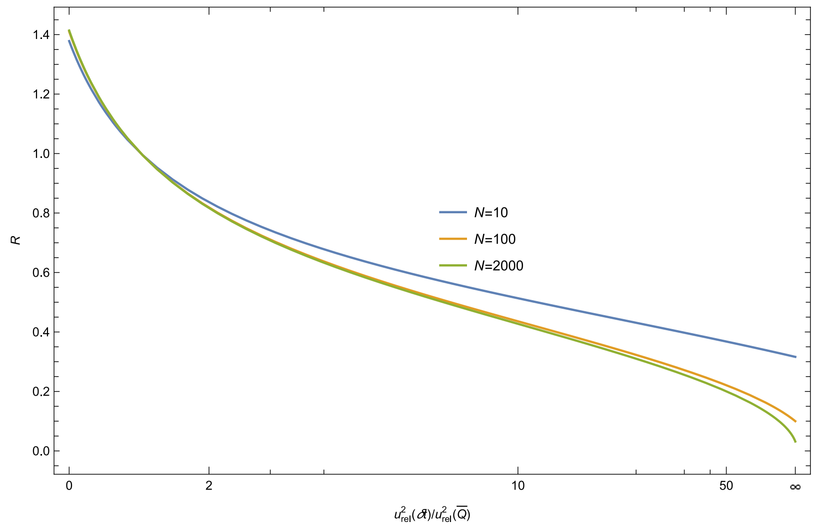


**Fig. 1.** The ordinate is $R = u_{\mathrm{rel}}(V_t)/u^*_{\mathrm{rel}}(V_t)$, the ratio of the delivered volume relative uncertainty calculated taking covariances into account to the same as currently calculated. The abscissa is the ratio of the squared relative uncertainty of the average flowrates to that of the time intervals.

Noting that the first term in the rightmost-hand side of expression (21) has the same structure as the first of expressions (9), one easily obtains

$$\boldsymbol{\delta t}^{\top} \boldsymbol{U}_{\overline{\boldsymbol{Q}}} \boldsymbol{\delta t} = \frac{u_{\mathrm{r}}^2}{2} \sum_{i=1}^{N} \delta t_i^2 + \frac{u_{\mathrm{r}}^2}{2} \sum_{i=1}^{N-1} \delta t_i \delta t_{i+1} + 4 u_{\mathrm{c}}^2 \Delta t^2. \tag{23}$$

By further considering that the time intervals are nominally equal and adopting the same assumption that led to Eq. (14), that is, that the flowrate is approximately constant,

$$u^2(V_t) \approx \overline{Q}^2 u_{\delta t}^2 + (2N-1)\,\delta t^2 \frac{u_{\mathrm{r}}^2}{2} + \Delta t^2 4 u_{\mathrm{c}}^2. \tag{24}$$

This expression gives the uncertainty associated with the indicated volume in the case of correlated flowrate indications, to be compared with Eq. (14). As the second term in the right-hand side is essentially the flowrate-indication uncertainty of Eq. (14), the uncertainty in the indicated volume in this case differs from that in the uncorrelated case by the term $\Delta t^2 4 u_{\mathrm{c}}^2$, which accounts for the traceability to the SI unit enabled by the calibration.

## 4. Example – Calibration of a static gas meter

In this section we consider the calibration of a static gas meter and compare the results provided by our uncertainty analysis against those yielded by the current practice.

For a given delivered volume, the flowmeter performance is strongly dependent on the flowrate, so that the calibration is carried out at several different flowrates, in order to cover the whole measuring range. For each flowrate, a reference flow nominally stable over the acquisition time is used as stimulus, and the instrument responses are taken using the gas meter in the so-called test mode, in which the sampling rate ($300\,\mathrm{ms} < \delta t_i < 600\,\mathrm{ms}$) is higher than in measuring mode ($\delta t_i \approx 2000\,\mathrm{ms}$)[3].

We calibrated a domestic gas meter of size G4 (maximum flowrate $6\,\mathrm{m}^3/\mathrm{h}$, *i.e.* $100\,\mathrm{L/min}$) against the INRiM bell prover [18]. The calibration was carried out at 8 different flowrates, ranging from $1\,\mathrm{L/min}$ to $100\,\mathrm{L/min}$.[4][5] The prover has an available volume of $120\,\mathrm{L}$ and, due to the characteristic of the measurement rig and in order to minimize its contribution to calibration uncertainty, the acquisition intervals $\Delta t$ are subjected to the double constraint of a minimum value $\Delta t_{\mathrm{min}} \approx 1\,\mathrm{min}$ and a minimum delivered volume $V_{t\mathrm{min}} \approx 20\,\mathrm{L}$.

For each flowrate we calculated the (estimated) delivered volume $V_t$ using Eq. (4). We then calculated the associated (relative standard) uncertainty according to Eq. (8) and compared it with that obtained from Eq. (13), *i.e.* using the current treatment neglecting covariances.

### 4.1. Input uncertainties

The input uncertainties to the measurement model are $u_{\overline{Q}}$ and $u_{\delta t}$.

As concerns $u_{\overline{Q}}$, it is made essentially of two contributions: the flowmeter finite resolution and the variability of its indications when submitted to a (nominally) stable stimulus.

We found experimentally the flowmeter resolution to be $\approx 3.9\,\mathrm{mL/min}$. The corresponding standard uncertainty is thus $u_{\mathrm{res}}(Q) \approx 1.13\,\mathrm{mL/min}$, that is, $\approx 0.1\,\%$ of $1\,\mathrm{L/min}$, the lowest flowrate allowed by our calibration rig, and negligible for higher flowrates.

The uncertainty component due to the indication stability, $u_{\mathrm{stab}}(Q)$, was calculated, at each flowrate, as the sample standard deviation of the indicated $Q_i$, $u_{\mathrm{stab}}(Q) = s(Q_i)$. It is to be noted that the stimulus is only nominally stable, and contributes an estimated $10\,\%$ to the total dispersion.

In conclusion, $u_Q^2 = u_{\mathrm{stab}}^2(Q) + u_{\mathrm{res}}^2(Q)$, and $u_{\overline{Q}} = u_Q/\sqrt{2}$.

The uncertainty $u_t$ is the same for all $t_i$ and comes from the resolution of the time counter of the gas meter, which is $1\,\mathrm{ms}$, so that $u_t = 0.29\,\mathrm{ms}$ and $u_{\delta t} = 0.41\,\mathrm{ms}$. This discretization uncertainty is non-negligible when the flowmeter is in the test mode (in our case $\delta t \approx 490\,\mathrm{ms}$, so that $u_{\mathrm{rel}}(\delta t) \approx 0.083\,\%$) and actually could impair the presumed advantage of a shorter interval in test mode. Put in a different way, this mode provides more time intervals but their duration is more uncertain.

### 4.2. Results

Table 1 gives, for each nominal reference flowrate $Q_{\mathrm{nom}}$ (column 1), acquisition interval $\Delta t$, number $N+1$ of indications $\{Q_i,\ t_i\}$ and relative standard uncertainty of the indicated average flowrate $u_{\mathrm{rel}}(\overline{Q})$

[3] A longer sampling period is used to reduce the power consumption of the instrument, which must work for at least 8 years with the original battery; notice that other energy-saving features, like the sleep/awake function, which means that the instrument is active only when needed, could also influence the measurement.

[4] As per our procedures, all measurements were repeated three times, but only one of the series is analyzed here.

[5] According to the customary practice, we use throughout litre and minute rather than cubic decimeter and second.

**Table 1**
Uncertainty evaluation for a gas meter calibration.

| $Q_{\mathrm{nom}}/(\mathrm{L/min})$ | $\Delta t/\mathrm{s}$ | $N+1$ | $u_{\mathrm{rel}}(\overline{Q})/\%$ | $V_t/\mathrm{L}$ | $u_V/\mathrm{mL}$ | $u_{\mathrm{rel}}(V_t)/‰$ | $u^*_{\mathrm{rel}}(V_t)/‰$ |
|---|---|---|---|---|---|---|---|
| 1 | 1204.280 | 2448 | 1.01 | 19.8003 | 5.8 | 0.29 | 0.21 |
| 5 | 248.703 | 502 | 0.43 | 20.5857 | 5.6 | 0.27 | 0.20 |
| 10 | 134.000 | 276 | 0.19 | 21.4112 | 3.5 | 0.17 | 0.13 |
| 20 | 57.312 | 121 | 0.20 | 18.3182 | 4.7 | 0.25 | 0.20 |
| 30 | 70.171 | 141 | 0.28 | 33.5666 | 11.4 | 0.34 | 0.25 |
| 50 | 73.375 | 152 | 0.32 | 58.4948 | 21.5 | 0.37 | 0.27 |
| 75 | 54.015 | 114 | 0.49 | 65.3673 | 43.0 | 0.66 | 0.48 |
| 100 | 52.769 | 110 | 0.45 | 84.6090 | 51.2 | 0.61 | 0.44 |

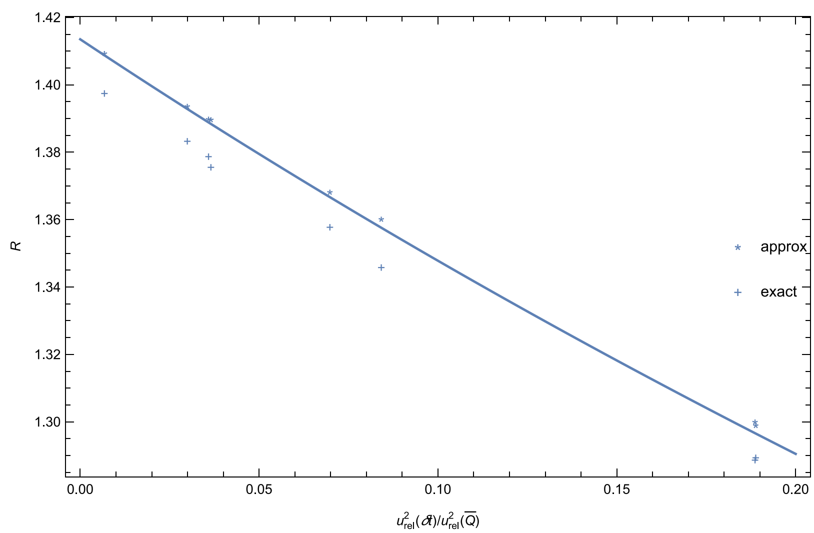


**Fig. 2.** Values of $R = u_{\mathrm{rel}}(V_t)/u^*_{\mathrm{rel}}(V_t)$ with the data of Table 1 (crosses). Asterisks denote values calculated with expression (18) (based on simplifying assumptions).

(columns 2 to 4). Columns 5 and 6 give volume $V_t$ and associated standard uncertainty obtained from Eqs. (4) and (8), respectively.

Columns 7 and 8 show the relative uncertainties associated with the estimates of the delivered volume, calculated using expressions (8) and (13), respectively. In both cases, the increases towards the extremes of the measuring range are due to limited resolution at low flowrates and to limited measuring time at high flowrates. The decrease at 100 L/min is consistent with a statistical fluctuation.

Fig. 2 shows the values of the ratio $R = u_{\mathrm{rel}}(V_t)/u^*_{\mathrm{rel}}(V_t)$ obtained from the data of Table 1 (crosses). Asterisks denote data calculated from expression (18), which is based on approximate assumptions, and are given here just to demonstrate the heuristic value of that expression.

## 5. Comments

In our example, the indications $Q_i$, as well as the $t_i$, are assumed to be uncorrelated, as in Section 3.1. Yet, even assuming uncorrelated indications, the uncertainty evaluated taking into account the covariances arising from the discrete-time integration is 30 % to 40 % greater than the uncertainty calculated neglecting them.

Of course, Eq. (8), used to obtain the results, can accommodate any matrix structure, and is therefore appropriate in any more-general situation, such as that discussed in Section 3.3.

Therefore, our approach is suitable in those situation in which the flow is highly unstable, so that the flowrate changes wildly and abruptly, which is typically the case when static gas meters are used online.

A correct uncertainty evaluation, *per se* desirable, is of paramount importance in the gas trade among countries, where the volumes delivered trough large offshore pipelines are measured by both parties and must be mutually consistent within stipulated limits. Given the immense amounts of money involved, it is evident that uncertainties play a key role, so that they should be evaluated according to an agreed, technically sound procedure.

But even for domestic gas meters, which usually are calibrated only once, before being installed online, to the purpose of checking the conformity to regulations establishing a maximum permissible error, uncertainty plays a role in that it impacts on consumer's risk [10].

## 6. Conclusions

We discuss in this paper the evaluation of uncertainty in the measurement of the volume of delivered gas by means of static meters. In such cases, the delivered volume is a function of flowrate and time, and the measurement model involves discrete-time integration of many average flowrates determined from subsequent 'instantaneous' flowrate indications.

It is intuitive that, even assuming the individual flowrate indications to be uncorrelated, any two contiguous average flowrates must be correlated, because they both are function of the middle indication, so that the uncertainty associated with the latter affects them both in the same way. For the same reason, any two adjacent time intervals must be correlated.

Nevertheless, to the best of our knowledge, this intuitive feature of the very mechanism of the discrete-time integration has been so far disregarded; from the (scarce) literature that it was possible to find, it appears that the average flowrates, as well as the corresponding time intervals, are considered as independent in the data analysis currently adopted in the industrial sectors applying such method.

In our paper we therefore develop a rigorous treatment in which the correlations arising between the aforementioned quantities are appropriately taken into account. We show that the current approach is conceptually flawed, and can lead to severe underrating (and possibly overrating) of the uncertainty associated with the estimate of the delivered volume. We also corroborate our findings with a practical example.

In a broader perspective, Eq. (8) works equally well for any quantity other than $Q$ for which a measurement model of the kind of expression (2), suitably expressed in matrix terms by formula (4), applies. In other words, Eq. (8) can (and, in our opinion, should) be adopted in any measurement in which the measurand estimate is obtained through time integration from a sensor which provides indications at discrete moments in time.

## CRediT authorship contribution statement

**Pier Giorgio Spazzini:** Conceptualization, Formal analysis, Software, Writing – original draft, Review and editing. **Walter Bich:** Conceptualization, Formal analysis, Software, Writing – original draft, Review and editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

Data will be made available on request.

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